MATHEMATICS - I

(Calculus and Linear Algebra)

For Computer Science Engineering Branches

Reena Garg

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by Reena Garg
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FOREWORD

Engineering has played a very significant role in the progress and expansion of mankind and society for centuries. Engineering ideas that originated in the Indian subcontinent have had a thoughtful impact on the world.

All India Council for Technical Education (AICTE) had always been at the forefront of assisting Technical students in every possible manner since its inception in 1987. The goal of AICTE has been to promote quality Technical Education and thereby take the industry to a greater heights and ultimately turn our dear motherland India into a Modern Developed Nation. It will not be inept to mention here that Engineers are the backbone of the modern society - better the engineers, better the industry, and better the industry, better the country.

NEP 2020 envisages education in regional languages to all, thereby ensuring that each and every student becomes capable and competent enough and is in a position to contribute towards the national growth and development.

One of the spheres where AICTE had been relentlessly working from last few years was to provide high-quality moderately priced books of International standard prepared in various regional languages to all it's Engineering students. These books are not only prepared keeping in mind it's easy language, real life examples, rich contents and but also the industry needs in this everyday changing world. These books are as per AICTE Model Curriculum of Engineering & Technology – 2018.

Eminent Professors from all over India with great knowledge and experience have written these books for the benefit of academic fraternity. AICTE is confident that these books with their rich contents will help technical students master the subjects with greater ease and quality.

AICTE appreciates the hard work of the original authors, coordinators and the translators for their endeavour in making these Engineering subjects more lucid.

(Anil D. Sahasrabudhe)

Acknowledgement

The author grateful to AICTE for their meticulous planning and execution to publish the technical book for Engineering and Technology students.

I sincerely acknowledge the valuable contributions of the reviewer of the book Prof. Garima Singh, for making it students' friendly and giving a better shape in an artistic manner.

This book is an outcome of various suggestions of AICTE members, experts and authors who shared their opinion and thoughts to further develop the engineering education in our country.

It is also with great honour that I state that this book is aligned to the AICTE Model Curriculum and in line with the guidelines of National Education Policy (NEP) -2020. Towards promoting education in regional languages, this book is being translated in scheduled Indian regional languages.

Acknowledgements are due to the contributors and different workers in this field whose published books, review articles, papers, photographs, footnotes, references and other valuable information enriched us at the time of writing the book.

Finally, I like to express our sincere thanks to the publishing house, M/s. Khanna Book Publishing Company Private Limited, New Delhi, whose entire team was always ready to cooperate on all the aspects of publishing to make it a wonderful experience.

Reena Garg

Preface

Mathematics is a necessary avenue to scientific knowledge which opens new vistas of mental ability. Engineering mathematics offers a balance of theory and practice, which is intellectually stimulating. Learning the craft of applying mathematics to real world problems allow an Engineering student to find the solutions of the problem.

Calculus and Linear Algebra is intended mainly for undergraduate students of B.Tech **(CSE)** of 21st century with the aim to provide a sound understanding in the subject of mathematics.. This book is strictly aligned with AICTE model curriculum incorporating student centric and self-learning activities as per New National Education Policy based on **OBE** and **Bloom Taxonomy**. The topics are well organized to create interest among readers to study and apply the mathematical tools in engineering and science disciplines. The book mainly emphasizes on the practical applications of the concepts discussed in the units which will help the students to incorporate a deliberate focus on problem - solving skills.

The book consists of **5** units. For more understanding of the topic, a good number of relatively competitive problems are given at the end of each unit in the form of **short questions, HOTS, assignments, MCQs** and **know more**. **Practical/Projects/Activity** also given in each unit for enhancing the student's capability and to increase the feeling of team work. To clarify the subject, the text has been supplemented through **Notes, Observations** and **Remarks**. An attempt has been made to explain the topics through maximum use of geometries wherever possible.

Unit-1 deals with the application of derivatives, curvature, definite and improper integrals, Beta-Gamma functions with their properties,

Unit-2 is concerned to find the solution by using Rolle's theorem, Mean value theorem, Taylor's and Maclaurin's theorems, L'Hospital Rule and Maxima-minima for one variable.

Unit-3 deals with matrices, determinant, solution of linear system of equations with various methods, rank, Crammer's Rule, Gauss Elimination method and Gauss Jordan method with examples.

Unit-4 focuses on vector space, dependence, independence of vectors, basis, dimension, Inverse of a linear transformation, rank- nullity theorem, composition of linear maps with matrix associated with it.

Unit-5 discusses eigen values, eigenvectors, diagonalization, Inner product spaces, Gram-Schmidt orthogonalization and theorems based of symmetric and skew-symmetric matrices.

Mathematics is a subject that can be mastered only through hard work and practice. Practice is the only key word in the learning process of mathematics.

I hope this book will meet the requirements and expectations of all the engineering students. Although every care has been taken to avoid misprints and mistakes, yet it is difficult to claim perfection. I will gratefully receive and acknowledge every comment and suggestions from the teachers and the students leading to improvements in the text as well as in solved examples.

Reena Garg

Outcome Based Education

For the implementation of an outcome based education the first requirement is to develop an outcome based curriculum and incorporate an outcome based assessment in the education system. By going through outcome based assessments evaluators will be able to evaluate whether the students have achieved the outlined standard, specific and measurable outcomes. With the proper incorporation of outcome based education there will be a definite commitment to achieve a minimum standard for all learners without giving up at any level. At the end of the programme running with the aid of outcome based education, a student will be able to arrive at the following outcomes:

PO-1. **Engineering knowledge:** Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.

PO-2. **Problem analysis:** Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.

PO-3. **Design/development of solutions:** Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.

PO-4. **Conduct investigations of complex problems:** Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.

PO-5. **Modern tool usage:** Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.

PO-6. **The engineer and society:** Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.

PO-7. **Environment and sustainability:** Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.

PO-8. **Ethics:** Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.

PO-9. **Individual and team work:** Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.

PO-10. Communication: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.

PO-11. Project management and finance: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.

PO-12. Life-long learning: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

Course Outcomes

After completion of the course the students will be able to:

CO-1: Apply Differential and Integral Calculus to notion of curvature, Centre of curvature and evaluate improper integrals using correct mathematical limit notation. Apart from these applications they will have a basic understanding of Beta and Gamma Functions

CO-2: Examine the behaviour of function for a given interval and expansion of trigonometric and transcendental functions

CO-3: Formulate, analyse, solve and apply the concept of matrices on the problems based on linear system of equations and relate them with linear transformations.

CO-4: Classify linear Independence and linear dependence of vectors and explain the concepts of rank, basis and dimension of vector Space, in addition of this, also learn to composition of linear maps and association with matrices.

CO-5: Apply essential tool to solve numerical problems based on Eigen values, Eigen vectors, Eigenbases, diagonalisation and orthogonalisation with the help of, linear algebra. Also deal with various properties of Eigen values which are used to solve many complex problems in various branches of engineering. In addition to that aware with the concept of norm of a vector , orthonormal and orthogonal vectors

Mapping of Course Outcomes with Programme Outcomes to be done according to the matrix given below:

Course Outcome	Expected Mapping with Programme Outcomes *(1- Weak Correlation; 2- Medium correlation; 3- Strong Correlation)*											
	PO-1	**PO-2**	**PO-3**	**PO-4**	**PO-5**	**PO-6**	**PO-7**	**PO-8**	**PO-9**	**PO-10**	**PO-11**	**PO-12**
CO-1	3	2	2	1	1	-	2	-	-	-	-	-
CO-2	3	2	2	2	-	-	-	-	-	-	-	1
CO-3	3	3	3	1	2	2	-	-	1	1	-	1
CO-4	3	2	1	1	1	1	-	-	-	-	-	-
CO-5	3	2	2	2	2	1	-	-	-	-	1	-

Abbreviations and Symbols

SYMBOLS AND FORMULAE

1. Number System

N	–	set of natural numbers
$\mathbb{Z}$	–	set of integers
Q	–	set of rational numbers
I	–	set of irrational numbers
$\mathbb{R}$	–	set of real numbers
C	–	set of complex numbers
R^n	–	set of n-tuples

2. Greek Letters

α	–	alpha
β	–	beta
γ	–	gamma
Γ	–	capital gamma
δ	–	delta
Δ	–	capital delta
ε	–	epsilon
ι	–	iota
θ	–	theta
λ	–	lambda
μ	–	mu
ϕ	–	phi
ψ	–	psi
η	–	eta
π	–	pi
ρ	–	rho
κ	–	kappa

3. Notation in sets

$\in$	–	belongs to
$\notin$	–	not belongs to
$\cup$	–	Union
$\cap$	–	Intersection
( )	–	open interval
[ ]	–	close interval
$\subseteq$	–	subset
$\not\subseteq$	–	not subset
$\subset$	–	proper subset
$\not\subset$	–	not a proper subset
$\supset$	–	superset
{ }	–	set
ϕ	–	empty set
$<$	–	strictly less than
$>$	–	strictly greater than
$\leq$	–	less than or equal to
$\geq$	–	greater than or equal to

4. Some Other Useful Symbols

$\sim$	–	equivalent to
$\leftrightarrow$	–	interchange
∞	–	infinity
$\int$	–	integral
!	–	factorial
$\Rightarrow$	–	implies
$\forall$	–	for all
$\Leftrightarrow$	–	implies and implied by
$\|\ \|$	–	norm
$\|\ \|$	–	modulus
:	–	colon
;	–	semicolon

$[A : B]$ or $[A/B]$ – Augmented Matrix

5. Nature of Roots of an Quadratic equations

If $ax^2 + bx + c = 0$ is quadratic, then

(*a*) its roots are given by $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(*b*) the sum of the roots is equal to $-b/a$

(*c*) product of the roots is equal to c/a

(*d*) $b^2 - 4ac = 0 \Rightarrow$ the roots are equal

(*e*) $b^2 - 4ac > 0 \Rightarrow$ the roots are real and distinct

(*f*) $b^2 - 4ac < 0 \Rightarrow$ the roots are complex

(*g*) If $b^2 - 4ac$ is a perfect square, then the roots are rational.

6. **Properties of Logarithm**

(*a*) $\log_a 1 = 0, \log_a 0 = -\infty$ for $a > 1$,

$\log_a a = 1$

$\log_e 2 = 0.6931$

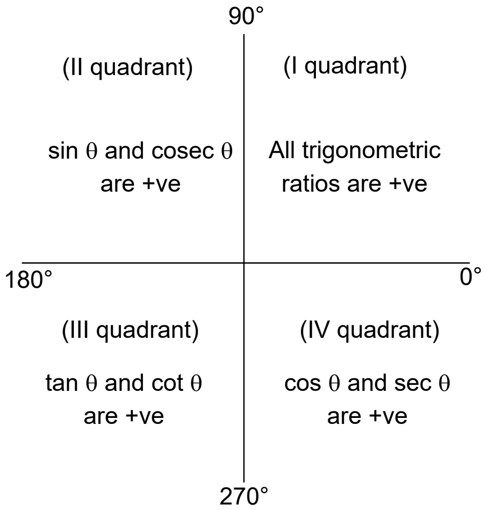
$\log_e 10 = 2.3026, \log_{10} e = 0.4343$

(*b*) $\log_a p + \log_a q = \log_a pq$

(*c*) $\log_a p - \log_a q = \log_a \dfrac{p}{q}$

(*d*) $\log_a p^q = q \log_a p$

7. **Nature of Trigonometric Ratios in Quadrant**



8. **Product and Sum Formulae for trigonometric functions**

(*a*) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

(*b*) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

(*c*) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(*d*) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(*e*) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(*f*) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(*g*) $\sin 2A = 2 \sin A \cos A = \dfrac{2 \tan A}{1 + \tan^2 A}$

(*h*) $\cos 2A = \cos^2 A - \sin^2 A$

$= 1 - 2\sin^2 A$

$= 2\cos^2 A - 1 = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

(*i*) $\tan 2A = \dfrac{\sin 2A}{\cos 2A} = \dfrac{2 \tan A}{1 - \tan^2 A}$

(*j*) $\sin 3A = 3 \sin A - 4 \sin^3 A$

(*k*) $\cos 3A = 4 \cos^3 A - 3 \cos A$

(*l*) $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3\tan^2 A}$

(*m*) $\sin A + \sin B = 2 \sin \dfrac{A + B}{2} \cos \dfrac{A - B}{2}$

(*n*) $\sin A - \sin B = 2 \cos \dfrac{A + B}{2} \sin \dfrac{A - B}{2}$

(*o*) $\cos A + \cos B = 2 \cos \dfrac{A + B}{2} \cos \dfrac{A - B}{2}$

(*p*) $\cos A - \cos B = 2 \sin \dfrac{A + B}{2} \sin \dfrac{B - A}{2}$

(*q*) $\sin A \cos B = \dfrac{1}{2} [\sin(A + B) + \sin(A - B)]$

(*r*) $\cos A \sin B = \dfrac{1}{2} [\sin(A + B) - \sin(A - B)]$

(*s*) $\cos A \cos B = \dfrac{1}{2} [\cos(A + B) + \cos(A - B)]$

(*t*) $\sin A \sin B = \dfrac{1}{2} [\cos(A - B) - \cos(A + B)]$

(*u*) $\sin x = 0 \Leftrightarrow x = n\pi, n \in Z$

(*v*) $\sin x = \pm 1 \Leftrightarrow x = (4n \pm 1) \dfrac{\pi}{2}, n \in Z$

(*w*) $\cos x = 0 \Leftrightarrow x = (2n + 1) \dfrac{\pi}{2}, n \in Z$

(*x*) $\cos x = \pm 1 \Leftrightarrow x = 2n\pi$ and $x = (2n + 1)\pi$, $n \in Z$

(*y*) $e^{ax} \neq 0, \forall x \in R; a \in R$

9. **Basic differentiation formulae**

(*a*) $\dfrac{d}{dx}(\sin x) = \cos x$

(*b*) $\dfrac{d}{dx}(\cos x) = -\sin x$

(*c*) $\dfrac{d}{dx}(\tan x) = \sec^2 x$

(*d*) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(*e*) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(*f*) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

(*g*) $\frac{d}{dx}(e^x) = e^x$

(*h*) $\frac{d}{dx}(a^x) = a^x \log_e a$

(*i*) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log a}$

(*j*) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(*k*) $\frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1}$

(*l*) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, x \neq \pm 1$

(*m*) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, x \neq \pm 1$

(*n*) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(*o*) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

(*p*) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}, x \neq 0, \pm 1$

(*q*) $\frac{d}{dx}(\text{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}, x \neq 0, \pm 1$

(*r*) $\frac{d}{dx}(\sin hx) = \cos hx$

(*s*) $\frac{d}{dx}(\cos hx) = -\sin hx$

10. Basic Integration Formulae

(*a*) $\int \sin x \, dx = -\cos x + c$

(*b*) $\int \cos x \, dx = \sin x + c$

(*c*) $\int \tan x \, dx = -\log \cos x + c = \log \sec x + c$

(*d*) $\int \cot x \, dx = \log \sin x + c$

(*e*) $\int \sec x \, dx = \log (\sec x + \tan x) + c$

(*f*) $\int \text{cosec } x \, dx = \log (\text{cosec } x - \cot x) + c$

(*g*) $\int \sec^2 x \, dx = \tan x + c$

(*h*) $\int \text{cosec}^2 x \, dx = -\cot x + c$

(*i*) $\int e^x dx = e^x$

(*j*) $\int a^x dx = \frac{a^x}{\log_e a} + c; a > 0, a \neq 1$

(*k*) $\int \frac{1}{x} dx = \log_e x + c$

(*l*) $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

(*m*) $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

(*n*) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log \left(\frac{a+x}{a-x} \right) + c$

(*o*) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \left(\frac{x-a}{x+a} \right) + c$

(*p*) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + c$

(*q*) $\int \frac{dx}{\sqrt{a^2+x^2}} = \sin h^{-1} \frac{x}{a} + c$

(*r*) $\int \frac{dx}{\sqrt{x^2-a^2}} = \cos h^{-1} \frac{x}{a} + c$

(*s*) $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$

(*t*) $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx)$

ABBREVIATIONS

lim	–	limit
$\therefore$	–	therefore
$\because$	–	because of
i.e.,	–	that is
$f^n(a)$	–	*n*th derivative of (f) at 'a'
sup.	–	supremum
Inf.	–	infimum
$Lf'(a)$	–	left hand derivative of 'f' at 'a'
$Rf'(a)$	–	right hand derivative of 'f' at 'a'
$Lf(a)$	–	left hand limit of 'f' at 'a'
$Rf(a)$	–	right hand limit of 'f' at 'a'
diag.	–	diagonal
L.H.S.	–	left hand side
R.H.S.	–	right hand side
dim	–	dimension
adj (A)	–	adjoint of matrix A
min.	–	minimum
max.	–	maximum
L.C.	–	linear combination
L.D.	–	linear dependence
L.I.	–	linear independence

List of Figures

Guidelines for Teachers

To implement Outcome Based Education (OBE) knowledge level and skill set of the students should be enhanced. Teachers should take a major responsibility for the proper implementation of OBE. Some of the responsibilities (not limited to) for the teachers in OBE system may be as follows:

- Within reasonable constraint, they should manipulate time to the best advantage of all students.
- They should assess the students only upon certain defined criterion without considering any other potential ineligibility to discriminate them.
- They should try to grow the learning abilities of the students to a certain level before they leave the institute.
- They should try to ensure that all the students are equipped with the quality knowledge as well as competence after they finish their education.
- They should always encourage the students to develop their ultimate performance capabilities.
- They should facilitate and encourage group work and team work to consolidate newer approach.
- They should follow Blooms taxonomy in every part of the assessment.

Bloom's Taxonomy

Level	Teacher should Check	Student should be able to	Possible Mode of Assessment
Creating	Students ability to create	Design or Create	Mini project
Evaluating	Students ability to Justify	Argue or Defend	Assignment
Analysing	Students ability to distinguish	Differentiate or Distinguish	Project/Lab Methodology
Applying	Students ability to use information	Operate or Demonstrate	Technical Presentation/ Demonstration
Understanding	Students ability to explain the ideas	Explain or Classify	Presentation/Seminar
Remembering	Students ability to recall (or remember)	Define or Recall	Quiz

Guidelines for Students

Students should take equal responsibility for implementing the OBE. Some of the responsibilities (not limited to) for the students in OBE system are as follows:

- Students should be well aware of each UO before the start of a unit in each and every course.
- Students should be well aware of each CO before the start of the course.
- Students should be well aware of each PO before the start of the programme.
- Students should think critically and reasonably with proper reflection and action.
- Learning of the students should be connected and integrated with practical and real life consequences.
- Students should be well aware of their competency at every level of OBE.

Contents

1 Calculus I

UNIT SPECIFICS

This unit elaborately discusses about the topics curvature, radius of curvature, centre of curvature, circle of curvature, evolutes, involute, envelope, definite and improper integrals, beta and gamma functions and their properties, applications of definite integrals to evaluate surface areas and volumes of revolutions. All the above topics have been discussed with ample examples so as to make theory application crystal clear to the students. Figures also included wherever required to make students visualize the topics.

RATIONALE

Involute and Evolute is a part of Differential geometry,which is itself a very important concept for students who are working in the field of Science, Artificial Intelligence and Robotics. It has many applications in day-to-day real life also.

One of the major application of Involute of circle is in designing of gears for revolving parts where gear tooth follow the shape of involute.

The basic application of involute usage is in winding clocks and toys where a winding key is used to motion the spiral spring in a circular involute.

We use definite integrals to calculate the force exerted on the dam when the reservoir is full and we examine how changing water levels affect that force. Indefinite integrals are used to find areas and volumes of curves of bounded bodies.

Gamma function is used in gamma distribution which is used to determine time based occurrence, such as life span of an electronic component.

PRE-REQUISITES

1. Basic knowledge of integration and differentiation.
2. Understanding of different curves like circle, ellipse, hyperbola etc.
3. Familiar with the concept of factorial.
4. Use integration to find out the area enclosed by two or more curves.

UNIT OUTCOMES

After completion of this unit, students will be able to:

U1-01: Explain the concept of curvature and radius of curvature; also find the evolutes of curve with the help of centre of curvature.

U1-02: Apply integral test on various functions to find their nature in terms of convergence and divergence.

U1-03: Familiarise themselves with the concept of Beta-Gamma functions and apply these to evaluate various integrals.

U1-04: Evaluate the surface area and the volume of solids of revolution for cartesian, parametric and polar curves.

MAPPING OF UNIT OUTCOMES WITH COURSE OUTCOMES

Unit 1 Outcomes	EXPECTED MAPPING WITH COURSE OUTCOMES (1- Weak Correlation; 2- Medium Correlation; 3- Strong Correlation)				
	CO-1	CO-2	CO-3	CO-4	CO-5
U1-01	3	1	–	–	1
U1-02	2	3	–	–	–
U1-03	3	–	–	–	–
U1-04	2	–	–	–	1

HISTORY

Apollonius (c. 262–190 BC) "calculated" curvature of conic sections implicitly when solving the problem of drawing normals to them in book V of Conica, but he did not think of it as a property of a curve, and his "calculations" are constructions of segments. The first person to "see" curvature was Oresme (c. 1320-1382), Descartes's precursor in introducing coordinates. He described it as a local measure of curve's bending, and christened it with the Latin "curvitas". Later he proposed that for circles it can be quantified by the reciprocal of the radius, our modern convention. Kepler vaguely suggested how to define curvature for general curves by considering the "closest" circle at a point, named osculating circle by Leibniz in 1680s. But it was Huygens, who first found a way to calculate curvature for general curves, and Newton who gave the concept its modern form.

"I believe that we do not know anything for certain everything probably."

—Christiaan Huygens

1.1 CURVATURE

In the Fig. 1.1, it can be seen that the given curve *AMNB* bends more sharply at the point *M* as comparison to the point *N*. The bending of a curve at a particular point is called the curvature of the curve at that point. So the curvature at *M* is more than the curvature at *N*. It will give a definite numerical measure of the sharpness of the bending of the curve at the point.

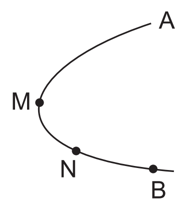


Fig. 1.1

In Fig. 1.2, let P be any point on a given curve and Q be a neighbouring point of P such that the arc PQ is concave towards its chord. Let the normals at P and Q intersects at N.

When $Q \to P$, N tends to a definite position C, called the centre of curvature of the curve at P. The distance CP is called the radius of curvature of the curve at the point P and is denoted be ρ (rho). The circle with centre at C and the radius B, is equal to CP, is called the circle of curvature of the given curve at the point P. Any chord of the circle at curvature drawn through the point P is called the chord of curvature. The reciprocal of the radius of curvature is called the curvature of the curve at the point P and is denoted by κ.

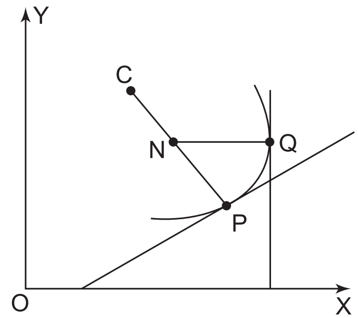


Fig. 1.2

1.1.1 Mathematical Definition of a Curvature

Let AB be a curve and P, Q be two neighbouring points on this curve. Let an arc $AP = s$ and the arc $AQ = s + \delta s$. 'A' is a fixed point on the curve from which arcs length are measured. Let the tangents at P and Q makes an angle ψ and $\psi + \delta\psi$ respectively with a fixed line, *i.e.*, x-axis, then

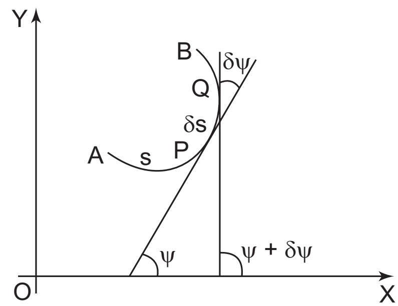


Fig. 1.3

i. The angle $\delta\psi$ through which the tangent turns as its point of the contact travels along the arc PQ is called the total bending or total curvature of the arc PQ.

ii. The ratio $\dfrac{\delta\psi}{\delta s}$ is called the mean or average curvature of the arc PQ.

iii. The limiting value of the mean curvature which $Q \to P$ is called the curvature of the curve at the point P.

Thus, the curvature (κ) at point P is $\underset{Q \to P}{\text{Lt}} \dfrac{\delta\psi}{\delta s} = \underset{\delta s \to 0}{\text{Lt}} \dfrac{\delta\psi}{\delta s} = \dfrac{d\psi}{ds}$.

iv. The reciprocal of the curvature of the curve at P, provided this curvature is not zero, is called the radius of curvature of the curve at P and is denoted by ρ *i.e.* $\rho = \dfrac{1}{\kappa} = \dfrac{ds}{d\psi}$.

Remarks: (1) A straight line does not bend at all (as ψ is constant, so $\dfrac{d\psi}{ds}$ is zero).

Hence curvature of a straight line is zero.

(2) Curvature of a circle is constant and equal to the reciprocal of its radius.

1.1.2 Radius of Curvature

The reciprocal of the curvature at any point is called the radius of curvature at that point. Obviously the curvature at any point should not be zero for defining radius of curvature at that point. It is usually denoted by ρ. Hence

$$\rho = \frac{1}{\kappa} = \frac{ds}{d\psi} \text{ at } P.$$

Graphically

For the curve CD, if ρ is the radius of curvature at point P, we draw normal at the point P and then $O'P$ is the distance equal to radius of curvature at P with O'.

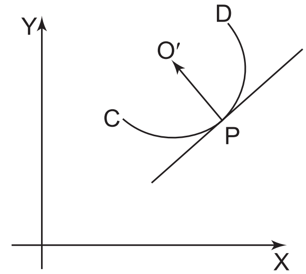


Fig. 1.4

A. Radius of Curvature for the Cartesian Curve

We find the expression for ρ, when the equation of curve is given in cartesian co-ordinates.

To find ρ for the curve $y = f(x)$

If ψ is the angle which the tangent at $P(x, y)$ on the curve makes with x-axis, then we have

$$\sin\psi = \frac{dy}{ds}, \cos\psi = \frac{dx}{ds} \text{ and } \tan\psi = \frac{dy}{dx}$$

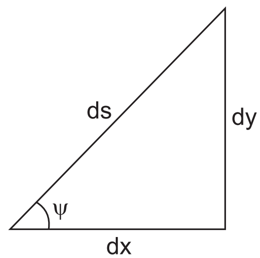


Fig. 1.5

So from the last relation, we have

$$\psi = \tan^{-1}(y_1) \text{ where } y_1 = \frac{dy}{dx}$$

Differentiating w.r.t x, we have

$$\frac{d\psi}{dx} = \frac{1}{1+y_1^2} \cdot y_2 \qquad \left\{\because\ y_2 = \frac{d^2y}{dx^2}\right\}$$

Now,
$$\frac{1}{\rho} = \frac{d\psi}{ds} = \frac{d\psi}{dx}\cdot\frac{dx}{ds} = \frac{y_2}{1+y_1^2}\cdot\cos\psi$$

$$= \frac{y_2}{1+y_1^2}\cdot\frac{1}{\sqrt{1+y_1^2}}$$

$$\left[\because\ \cos\psi = \frac{1}{\sec\psi} = \frac{1}{\sqrt{1+\tan^2\psi}} = \frac{1}{\sqrt{1+(dy/dx)^2}} = \frac{1}{\sqrt{1+y_1^2}}\right]$$

So,
$$\frac{1}{\rho} = \frac{y_2}{(1+y_1^2)^{3/2}}$$

$$\rho = \frac{[1+y_1^2]^{3/2}}{y_2} \text{ where } y_2 \neq 0$$

B. Radius of Curvature for Parametric Curve

To find ρ for the curve $x = f(t), y = \phi(t)$ *i.e.*, when the parametric equation of the curve is given.

Here $x = f(t)$ and $y = \phi(t)$, t being parameter

We know,
$$x' = \frac{dx}{dt}, y' = \frac{dy}{dt}$$

Also, $$x'' = \frac{d^2x}{dt^2}, y'' = \frac{d^2y}{dt^2}$$

Thus, $$y_1 = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'}{x'} \qquad ...(1)$$

Differentiating (1) w.r.t. x, we have

$$y_2 = \frac{x'y'' - y'x''}{x'^2} \cdot \frac{dt}{dx} = \frac{x'y'' - y'x''}{x'^3} \qquad \left[\because \frac{dt}{dx} = \frac{1}{x'}\right]$$

We know, $\rho = \frac{[1+y_1^2]^{3/2}}{y_2}$ so, after putting all values, we have

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$$

C. Radius of Curvature for Polar Curve

To find ρ for the curve $r = f(\theta)$ or $f(r, \theta) = 0$

Let ϕ be the angle which the tangent at $P(r, \theta)$ makes with OP, then we have

$$\tan\phi = \frac{rd\theta}{dr}, \quad \sin\phi = \frac{rd\theta}{ds} \quad \text{and} \quad \cos\phi = \frac{dr}{ds} \qquad ...(1)$$

Again if ψ be the angle which the tangent $P(r, \theta)$ makes with OX. then

$$\psi = \theta + \phi$$

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}$$

$$= \frac{d\theta}{ds} + \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds}$$

$$= \frac{d\theta}{ds}\left[1 + \frac{d\phi}{d\theta}\right] \qquad ...(2)$$

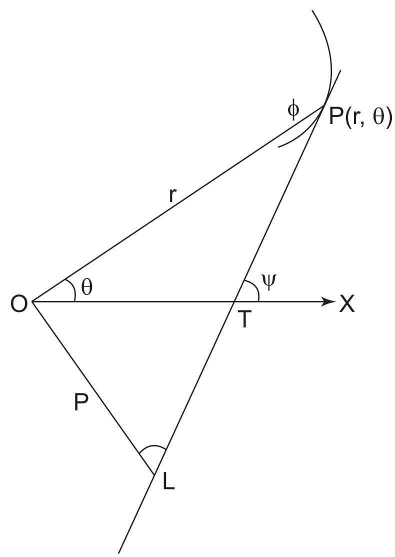


Fig. 1.6

From (1), $$\tan\phi = r \cdot \frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}}$$

or $$\tan\phi = \frac{r}{r_1} \quad \text{where } r_1 = \frac{dr}{d\theta}$$

Differentiating both side w.r.t θ, we have

$$\sec^2\phi \cdot \frac{d\phi}{d\theta} = \frac{r_1 \cdot r_1 - rr_2}{r_1^2}$$

$$\Rightarrow \quad \frac{d\phi}{d\theta} = \frac{r_1^2 - rr_2}{r_1^2} \cdot \frac{1}{\sec^2\phi}$$

$$= \frac{r_1^2 - rr_2}{r_1^2} \cdot \frac{1}{1+\tan^2\phi}$$

$$= \frac{r_1^2 - rr_2}{r_1^2} \cdot \frac{r_1^2}{r_1^2 + r^2} \qquad \left[\because \tan\phi = \frac{r}{r_1}\right]$$

$$= \frac{r_1^2 - rr_2}{r_1^2 + r^2} \qquad ...(3)$$

Again from (1) $\quad r\dfrac{d\theta}{ds} = \sin\phi = \dfrac{1}{\operatorname{cosec}\phi}$

$\Rightarrow \quad \dfrac{d\theta}{ds} = \dfrac{1}{r} \cdot \dfrac{1}{\sqrt{1+\cot^2\phi}} = \dfrac{1}{\sqrt{r^2 + r_1^2}} \qquad ...(4)$

From (2), (3) and (4), we have

$$\frac{d\psi}{ds} = \frac{1}{\sqrt{r^2 + r_1^2}} \cdot \left[1 + \frac{r_1^2 - rr_2}{r_1^2 + r^2}\right]$$

$$\frac{1}{\rho} = \frac{2r_1^2 + r^2 - rr_2}{(r^2 + r_1^2)^{3/2}} \qquad \left\{\because \frac{1}{\rho} = \frac{d\psi}{ds}\right\}$$

$\therefore \quad \rho = \dfrac{ds}{d\psi} = \dfrac{(r^2 + r_1^2)^{3/2}}{2r_1^2 + r^2 - rr_2}$

SOME SOLVED EXAMPLES

Example 1.1. *Find the radius of curvature at the given point of following curves:*

a. $y = 4\sin x - \sin 2x$ *at* $x = \dfrac{\pi}{2}$ b. $\sqrt{x} + \sqrt{y} = 1$ *at the point* $\left(\dfrac{1}{4}, \dfrac{1}{4}\right)$

Solution. (*a*) Given, $\quad y = 4\sin x - \sin 2x \qquad ...(1)$

Differentiate (1) w.r.t x

$$\frac{dy}{dx} = 4\cos x - 2\cos 2x$$

and $\quad \dfrac{d^2y}{dx^2} = -4\sin x + 4\sin 2x$

We know, $\quad \rho = \dfrac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}} = \dfrac{\left[1 + (4\cos x - 2\cos 2x)^2\right]^{3/2}}{-4\sin x + 4\sin 2x}$

$$\therefore \qquad \rho \text{ at } \frac{\pi}{2} = \frac{\left[1+\left(4\cos\frac{\pi}{2}-2\cos\pi\right)^2\right]^{3/2}}{-4\sin\frac{\pi}{2}} = \frac{(1+4)^{3/2}}{-4} \qquad \text{(ignoring the negative sign)}$$

$$= \frac{5^{3/2}}{4} \quad \textbf{(Answer)}$$

b. Given, $\sqrt{x}+\sqrt{y} = 1$ ...(1)

Differentiating the equation (1) w.r.t x

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{-\frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{y}}} = -\frac{\sqrt{y}}{\sqrt{x}}$$

$$= -1 \text{ at } \left(\frac{1}{4},\frac{1}{4}\right)$$

Similarly, we have

$$\frac{d^2y}{dx^2} = \frac{\left(\sqrt{x}\,.\,\frac{1}{2\sqrt{y}}\frac{dy}{dx}-\sqrt{y}\,.\,\frac{1}{2\sqrt{x}}\right)}{x}$$

$$= \frac{\left[\frac{\sqrt{x}}{2\sqrt{y}}\left(-\frac{\sqrt{y}}{\sqrt{x}}\right)-\frac{\sqrt{y}}{2\sqrt{x}}\right]}{x} = \frac{\left[-\frac{1}{2}-\frac{\sqrt{y}}{2\sqrt{x}}\right]}{x}$$

$$= -\left[\frac{\sqrt{x}+\sqrt{y}}{2x\sqrt{x}}\right] = \frac{-\left(\frac{1}{2}+\frac{1}{2}\right)}{\left(2\,.\,\frac{1}{4}\,.\,\frac{1}{2}\right)} \qquad \left(\text{at } \frac{1}{4},\frac{1}{4}\right)$$

$$= -4$$

We know,

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{-(1+1)^{3/2}}{4} = \frac{2^{3/2}}{4} = \frac{1}{\sqrt{2}} \quad \textbf{(Answer)}$$

Example 1.2. *Find the least value of* $|\rho|$ *for* $y = \log x,\ x > 0$.

Solution. Let $y = \log x$ ...(1)

Differentiating (1) w.r.t x, we have

$$\frac{dy}{dx} = \frac{1}{x} \text{ and } \frac{d^2y}{dx^2} = -\frac{1}{x^2}$$

We know, $$\rho = \frac{\left[1+y_1^2\right]^{3/2}}{y_2} = \frac{\left(1+\frac{1}{x^2}\right)^{3/2}}{-\frac{1}{x^2}}$$

$$= \frac{-(x^2+1)^{3/2}}{x}$$

Let $$|\rho| = f(x) = \frac{(x^2+1)^{3/2}}{x}$$

To find the least value of $f(x)$, we find $f'(x)$

$$f'(x) = \frac{x \cdot \frac{3}{2}(x^2+1)^{1/2}\, 2x - (x^2+1)^{3/2}}{x^2}$$

$$= \frac{3x^2\sqrt{x^2+1} - (x^2+1)^{3/2}}{x^2}$$

$$= \frac{\sqrt{x^2+1}\,(3x^2 - x^2 - 1)}{x^2} = \frac{\sqrt{x^2+1}\,(2x^2-1)}{x^2}$$

Equate $f'(x) = 0$, we get $x = \pm \frac{1}{\sqrt{2}}$

and also $f''(x)$ is positive for $x = \frac{1}{\sqrt{2}}$ (students can check)

Hence $|\rho|$ is minimum for $x = \frac{1}{\sqrt{2}}$ $(\because \quad x > 0)$

$\therefore$ $$|\rho|_{\min} = \left[\frac{(x^2+1)^{3/2}}{x}\right]_{x=\frac{1}{\sqrt{2}}} = \frac{\left(\frac{1}{2}+1\right)^{3/2}}{\frac{1}{\sqrt{2}}}$$

$$= \left(\frac{3}{2}\right)^{3/2}\sqrt{2} = \frac{3\sqrt{3}}{2}$$

$\therefore$ Minimum value of $|\rho|$ is $\frac{3\sqrt{3}}{2}$. **Answer**

Example 1.3. *Find the radius of curvature for Rectangular hyperbola $xy = c^2$ at the point (x, y).*

Solution. Hint: Take $y = \frac{c^2}{x}$. **Answer:** $\frac{(x^2+y^2)^{3/2}}{2c^2}$.

Example 1.4. *Find the radius of curvature at the origin of the two branches of the curve given by $x = 1 - t^2$, $y = t - t^3$.*

Solution. At origin (0, 0) two common value for t are 1 and –1. $[\because \; x = 1 - t^2,\ 0 = 1 - t^2$ at $x = 0$

Hence for two branches of the curves, value of t are 1 and –1 $\quad 1 = t^2$

$t = \pm 1]$

Given, $x = 1 - t^2$, $y = t - t^3$

Differentiating the above w.r.t '*t*', we get

$$\frac{dy}{dt} = 1 - 3t^2, \quad \frac{dx}{dt} = -2t$$

$$\frac{dy}{dx} = \frac{1-3t^2}{-2t} = -\frac{1}{2t} + \frac{3}{2}t \qquad ...(1)$$

Again differentiating (1) w.r.t. '*x*', we get

$$\frac{d^2y}{dx^2} = \left(\frac{1}{2t^2} + \frac{3}{2}\right) \cdot \frac{dt}{dx} = \frac{\left(\frac{1}{2t^2} + \frac{3}{2}\right)}{-2t}$$

$$= \frac{-1}{4t^3} - \frac{3}{4t}$$

We find, $\left(\frac{dy}{dx}\right)_{t=1} = -\frac{1}{2} + \frac{3}{2} = 1$

and $\left(\frac{d^2y}{dx^2}\right)_{t=1} = -\frac{1}{4} - \frac{3}{4} = -1$

Similarly, $\left(\frac{dy}{dx}\right)_{t=-1} = \frac{1}{2} - \frac{3}{2} = -1$

and $\left(\frac{d^2y}{dx^2}\right)_{t=-1} = \frac{1}{4} + \frac{3}{4} = 1$

So, $$(\rho)_{t=1} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1+1)^{3/2}}{-1} = -2\sqrt{2}$$

and $$(\rho)_{t=-1} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1+1)^{3/2}}{1} = 2\sqrt{2} \quad \textbf{(Answer)}$$

Example 1.5. *Prove that the radius of curvature for the catenary $y = c \cosh \frac{x}{c}$ is equal to the portion of the normal intercepted between the curve and the x-axis and that it varies as the square of the ordinate.*

Solution. Try yourself.

Example 1.6. *If ρ_1 and ρ_2 be the radii of curvature at the ends of focal chord of the parabola $y^2 = 4ax$, then prove that $\rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3}$.*

Solution. The equation of parabola is $y^2 = 4ax$, which in parametric form is $x = at^2, y = 2at$. If $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ are the two extremities of a focal chord of the parabola, then

$$t_1t_2 = -1$$

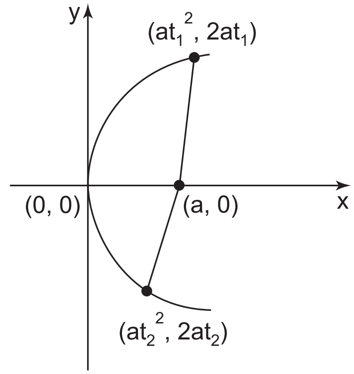


Fig. 1.7

For $\quad x = at^2, y = 2at$

We have, $\quad \dfrac{dx}{dt} = 2at, \dfrac{dy}{dt} = 2a$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{t}$

and $\quad \dfrac{d^2y}{dx^2} = \dfrac{-1}{t^2}\cdot\dfrac{dt}{dx} = -\dfrac{1}{2at^3}$

We know,
$$\rho = \frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}} = \frac{\left[1+\dfrac{1}{t^2}\right]^{3/2}}{-\dfrac{1}{2at^3}}$$

$$= -2a\ (t^2+1)^{3/2}$$

$\therefore \quad \rho \text{ at } (at_1^2, 2at_1) = -2a\ (t_1^2+1)^{3/2}$

and $\quad \rho \text{ at } (at_2^2, 2at_2) = -2a\,(t_2^2+1)^{3/2}$

$\therefore \quad (\rho_1)^{-2/3} + (\rho_2)^{-2/3} = \left[2a\,(t_1^2+1)^{3/2}\right]^{-2/3} + \left[2a\,(t_2^2+1)^{3/2}\right]^{-2/3}$

$$= (2a)^{-2/3}\left[\frac{1}{t_1^2+1}+\frac{1}{t_2^2+1}\right] = (2a)^{-2/3}\left[\frac{t_2^2+1+t_1^2+1}{(t_1^2+1)(t_2^2+1)}\right]$$

$$= (2a)^{-2/3}\left[\frac{t_1^2+t_2^2+2}{t_1^2+t_2^2+(-1)^2+1}\right] = (2a)^{-2/3}$$ **Proved.** $[\because\ t_1t_2 = -1]$

Example 1.7. *Show that the radius of curvature at the end of the major axis of an ellipse* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$ *is equal to the semi-latus rectum of the ellipse.*

Solution. Equation of an ellipse in parametric form is $x = a \cos t, y = b \sin t$
and the ends of major axis are $(\pm\, a, 0)$
Differentiating w.r.t. t, we get

$$\frac{dx}{dt} = -a\sin t,\ \frac{dy}{dt} = b\cos t$$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{-b}{a}\cot t$

and $\quad \dfrac{d^2y}{dx^2} = \dfrac{b}{a}\operatorname{cosec}^2 t\,\dfrac{dt}{dx} = \dfrac{b}{a}\operatorname{cosec}^2 t \times \dfrac{1}{-a\sin t}$

$$= \frac{-b}{a^2}\operatorname{cosec}^3 t$$

Radius of curvature is, $\rho = \dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}} = \dfrac{\left[1+\dfrac{b^2\cos^2 t}{a^2\sin^2 t}\right]^{3/2}}{\dfrac{-b}{a^2}\text{cosec}^3 t}$

$= \dfrac{1}{ab}(a^2\sin^2 t + b^2\cos^2 t)^{3/2}$ (ignoring the negative sign)

$= \rho$ at $(a, 0)$ is $\dfrac{1}{ab}(a^2\sin^2 0 + b^2\cos^2 0)^{3/2}$

$= \dfrac{b^2}{a}$ (Semi-latus rectum of the ellipse) **Proved.**

Example 1.8. *Find the radius of curvature for the curve* $x = c\log(s+\sqrt{s^2+c^2}), y = \sqrt{s^2+c^2}$

Solution. Given, $x = c\log(s+\sqrt{s^2+c^2})$

Differentiating w.r.t. s, we get

$$\frac{dx}{ds} = \frac{c}{s+\sqrt{s^2+c^2}}\left(1+\frac{2s}{2\sqrt{s^2+c^2}}\right) = \frac{c}{\sqrt{s^2+c^2}}$$

and $$\frac{dy}{ds} = \frac{2s}{2\sqrt{s^2+c^2}} = \frac{s}{\sqrt{s^2+c^2}}$$

So, $$\frac{dy}{dx} = \frac{\dfrac{dy}{ds}}{\dfrac{dx}{ds}} = \frac{s}{c} \qquad ...(1)$$

We know, $\tan\psi = \dfrac{dy}{dx}$

So, $s = c\tan\psi$ [from (1)]

$\therefore$ Radius of curvature, $\dfrac{ds}{d\psi} = c\sec^2\psi$

$= c(1+\tan^2\psi)$

$= c\left(1+\dfrac{s^2}{c^2}\right)$

$= \dfrac{c^2+s^2}{c}$ **(Answer)**

Example 1.9. *Prove that in the curve* $r^2 = a^2\sin 2\theta$*, the curvature varies as the radius vector.*

Solution. Try yourself.

Example 1.10. *If* ρ_1 *and* ρ_2 *are the radii of curvature at the extremities of any chord through the pole of the cardioid* $r = a(1-\cos\theta)$*, show that* $\rho_1^2+\rho_2^2 = \dfrac{16a^2}{9}$.

Solution. The cardioid $r = a(1 - \cos\theta)$ is as shown in Fig. 1.8.

If the point P_1 is (r_1, θ), then P_2 would be $(r_2, \theta + \pi)$ as P_1 and P_2 are the extremities of the chord through pole.

then, $\frac{dr}{d\theta} = r_1 = a\sin\theta$

and $\frac{d^2r}{d\theta^2} = r_2 = a\cos\theta$

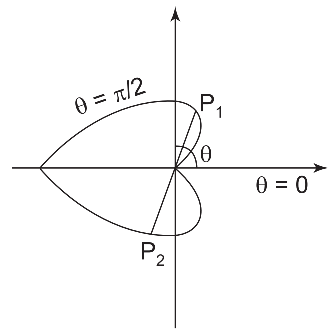


Fig. 1.8

then,
$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$
$$= \frac{[a^2(1-\cos\theta)^2 + a^2\sin^2\theta]^{3/2}}{a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a^2(1-\cos\theta)\cos\theta}$$
$$= \frac{a^3(1\cos\theta)^{3/2}\,2\sqrt{2}}{3a^2\,(1-\cos\theta)} = \frac{a}{3}2\sqrt{2}\,(1-\cos\theta)^{1/2}$$

$\therefore$
$$\rho_1 = \frac{2\sqrt{2}}{3}a(1-\cos\theta)^{1/2}$$

and,
$$\rho_2 = \frac{2\sqrt{2}}{3}a[1-\cos(\theta+\pi)]^{1/2}$$
$$= \frac{2\sqrt{2}}{3}a(1+\cos\theta)^{1/2}$$

$\therefore$
$$\rho_1^2 + \rho_2^2 = \frac{8a^2}{9}(1-\cos\theta+1+\cos\theta) = \frac{16a^2}{9}$$
$$\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}.$$ **Proved.**

EXERCISE 1.1

Find the radius of curvature at the given point of the following curves:

1. $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $\left(\frac{a}{4}, \frac{a}{4}\right)$

2. $x^3 + y^3 = 3axy$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

3. $y^2 = \frac{a^2(a-x)}{x}$ at $(a, 0)$ (**Hint:** Equation of curvature is $x = \frac{a^3}{y^2 + a^2}$)

4. $x^2y = a(x^2 + y^2)$ at $(-2a, 2a)$

5. Find the radius of curvature of the curve $y = e^x$ at the point where it crosses the y-axis.

6. Find the radius of curvature at any point $(0, c)$ of the catenary $y = c\cos h\,\frac{x}{c}$.

7. Show that for the parabola $y^2 = 4ax$, ρ^2 varies as $(SP)^3$, where ρ is the radius of curvature at any point P of the parabola and S is the focus of the parabola.

8. The tangent at two points P, Q on the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ are at right angles: show that if ρ_1, ρ_2 be the radii of curvature at these points, then $\rho_1^2 + \rho_2^2 = 16a^2$.

9. Prove that the radius of curvature at any point of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is three times the length of the perpendicular from the origin to the tangent at the point.

10. Prove that for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \rho=\frac{a^2b^2}{P^3}$ where P is the perpendicular from the center upon the tangent at (x, y).

11. Find the points on the parabola $y^2 = 8x$ at which the radius of curvature is $7\frac{13}{16}$.

12. Show that for the curve; $x = a \cos \theta\ (1 + \sin \theta)$, $y = a \sin \theta\ (1 + \cos \theta)$, the radius of curvature at $\theta = \frac{-\pi}{4}$ is a.

Find the radius of curvature for the following curves:

13. $r = a \cos n\theta$

14. $r^m = a^m \sin m\theta$

15. $r^2 \cos 2\theta = a^2$

16. Show that radius of curvature at any point of the curve $r = a \cos n\theta$, where $r = a$ is $\frac{a}{1+n^2}$.

17. Show that radius of curvature for the curve $\theta = \frac{\sqrt{r^2-a^2}}{a} - \cos^{-1}\frac{a}{r}$ is $r\sqrt{r^2-a^2}$.

18. Show that the radius of curvature of the lemniscate $r^2 = a^2 \cos 2\theta$ at the point where the tangent is parallel to x-axis is $\frac{\sqrt{2}}{3}a$.

Answers

1. $\frac{a}{\sqrt{2}}$	2. $\frac{3a}{8\sqrt{2}}$	3. $\frac{a}{2}$	4. 2
5. $2\sqrt{2}$	6. c	11. $\left(\frac{9}{8}, 3\right)$ and $\left(\frac{9}{8}, -3\right)$	
13. $\frac{(r^2+a^2n^2-r^2n^2)^{3/2}}{r^2-r^2n^2+2a^2n^2}$	14. $\frac{a^m}{(m+1)^{r^{m-1}}}$	15. $\frac{r^3}{a^2}$	

1.1.3 Centre of Curvature, Circle of Curvature

1.1.3.1 Centre of Curvature

The centre of curvature at any point P of a curve AB is the point which lies on the positive direction of the normal at P and is at a distance equal to the radius of curvature from it.

1.1.3.2 Circle of Curvature

If 'C' is the centre of curvature, then the circle with centre 'C' and radius of curvature 'ρ' passing through the point P, is called the circle of curvature.

Let ρ be the radius of curvature and $(\bar{x}, \bar{y})$ be the coordinates of the centre of curvature at a given point, then the equation of the circle of curvature is given by $(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$.

1.1.4 Coordinates of the Centre of Curvature

Let $(\bar{x}, \bar{y})$ be the coordinates of the centre of curvature C, lying on the normal at $P(x, y)$ on the curve such that $PC = \rho$.

From the figure 1.9, we have

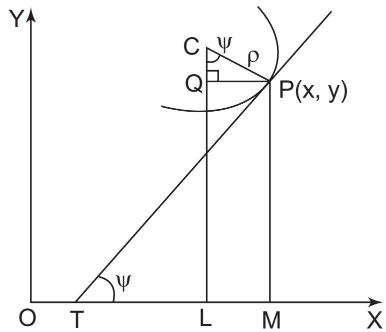


Fig. 1.9

$$\bar{x} = OL = OM - LM$$
$$= OM - PQ \qquad (\because \quad LM = PQ)$$

Now $\qquad OM = x$

$$PQ = PC \sin\psi = \rho \sin\psi$$

$\therefore \qquad \bar{x} = x - \rho\sin\psi \qquad \ldots(1)$

Similarly, $\qquad \bar{y} = CL = CQ + QL$

$$= CQ + PM \qquad (\because \quad QL = PM)$$

Now $\qquad CQ = PC\cos\psi = \rho\cos\psi$ and $PM = y$

$\therefore \qquad \bar{y} = y + \rho\cos\psi \qquad \ldots(2)$

As $\qquad \tan\psi = \dfrac{dy}{dx} = y_1$

$\therefore \qquad \sin\psi = \tan\psi \,.\, \cos\psi$

$$= \frac{\tan\psi}{\sec\psi} = \frac{\tan\psi}{\sqrt{1+\tan^2\psi}} \quad \text{or} \quad \sin\psi = \frac{y_1}{\sqrt{1+y_1^2}}$$

Using the above values of $\sin\psi$ and $\cos\psi$ in equations (1) and (2), we get

$$\bar{x} = x - \rho\frac{y_1}{\sqrt{1+y_1^2}} \quad \text{and} \quad \bar{y} = y + \frac{\rho}{\sqrt{1+y_1^2}}$$

But, as we know $\qquad \rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

So, we have, $\qquad \bar{x} = x - \dfrac{y_1}{y_2}(1 + y_1^2) \qquad \ldots(3)$

and $\qquad \bar{y} = y + \dfrac{1}{y_2}(1 + y_1^2) \qquad \ldots(4)$

are the required coordinates of the centre of curvature.

If we eliminate x, y between the equations (3) and (4) and the equation of the curve, we obtain a relation between $\bar{x}$ and $\bar{y}$ which is the equation of the evolute.

HISTORY

Huygens named the locus of the centers of curvature to a curve its evolute, and showed how to construct a perfect pendulum, whose period does not depend on its amplitude. The construction was based on the fact that evolute to a cycloid is congruent to it.

1.1.5 Evolute

Corresponding to each point on a curve we can find the curvature of the curve at that point. Drawing the normal at these points, we can find Centre of Curvature corresponding to each of these points. Since the curvature varies from point to point, centre's of curvature also differ. The totality of all such centres of curvature of a given curve will define another curve and this curve is called the evolute of the curve.

The Locus of centres of curvature of a given curve is called the evolute of that curve. The locus of the centre of curvature C of a variable point P on a curve is called the evolute of the curve. The curve itself is called involute of the evolute.

Here, for different points on the curve, we get different centre of curvatures. The locus of all these centres of curvature is called as Evolute. The external curve which satisfies all these centres of curvature is called as Evolute. Here Evolute is nothing but an curve equation.

To find Evolute, the following models exist.

If an equation of the curve is given and if we have to prove, L.H.S = R.H.S., then following steps should be followed:

1. First find Centre of Curvature, $C(\bar{x}, \bar{y})$ where $\bar{x} = x - [y_1(1 + y_1^2)]/y_2$, $\bar{y} = y + [(1 + y_1^2)]/y_2$, and then consider L.H.S: In that directly substitute $\bar{x}$ in place of x and $\bar{y}$ in place of y. Similarly for R.H.S. and then show that L.H.S = R.H.S.
2. If a curve is given and if we are asked to find the evolute of the given curve, then do as follows: First find Centre of curvature $C(\bar{x}, \bar{y})$ and then re-write as x in terms of $\bar{x}$ and y in terms of $\bar{y}$ and then substitute in the given curve, which gives us the required evolute.
3. If a curve is given, which is in parametric form, then first find Centre of curvature, which will be in terms of parameter. Then using these values of $\bar{x}$ and $\bar{y}$ eliminate the parameter, which gives us evolute.

Pictorial representation of Evolute and Centre of Curvature in Fig. 1.10.

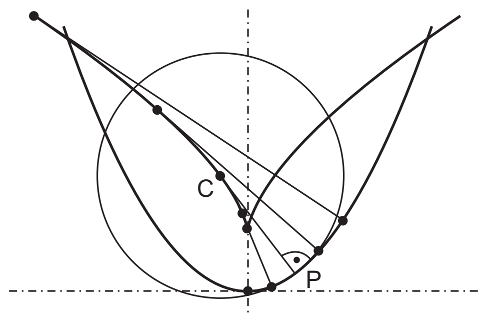


Fig. 1.10

1.1.6 Involute

A curve that is obtained by attaching a string which is imaginary and then winding and unwinding it tightly on the curve given is called involute in differential geometry. Involute or evolvent is the locus of the free end of this string.

For more Clarifications: The **evolute** of an involute of a curve is referred to that original curve. In other words, the locus of the center of curvature of a curve is called evolute and the traced curve itself is known as the involute of its evolute.

Remark: This is a part of a special branch of geometry called differential Geometry of Curves. It talks about the smooth curves which lie in Euclidean space and has applications of different methods of integral and differential calculus on them. The shapes related to some other curves are called involutes. This was discovered by Christine Huygens in 1673. He was a Dutch mathematician and a physicist.

1.1.6.1 Involutes of the Curves

Here we will see the involutes of the different curves as shown below:

- Involute of a Circle
- Involute of a Catenary
- Involute of a Deltoid
- Involute of a Parabola
- Involute of an Ellipse

1. **Involute of a Circle:** It is similar to the Archimedes spiral.

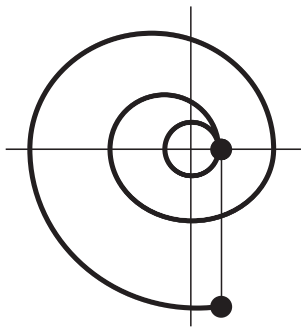


Fig. 1.11. Involute of a Circle

2. **Involute of a Catenary:** It is a curve which is similar to hanging cable supported by its ends. So, it is a *U* shaped hanging chain which looks like a parabola. The tractrix is the involute of the catenary through the vertex.

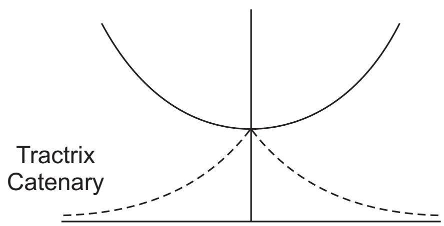


Fig. 1.12. Involute of a Catenary

3. **Involute of a Deltoid:** It is a tricuspid curve with three cusps. It resembles greek letter delta.

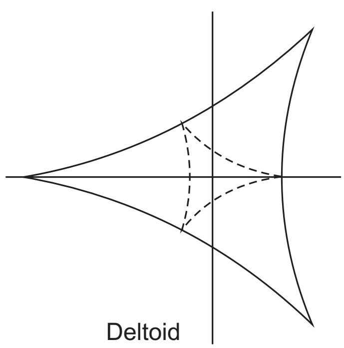


Fig. 1.13. Involute of a Deltoid

4. **Involute of a Parabola:**

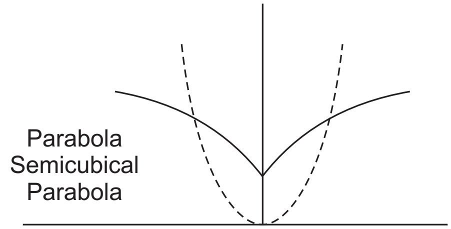


Fig. 1.14. Involute of a Parabola

5. **Involute of an Ellipse:**

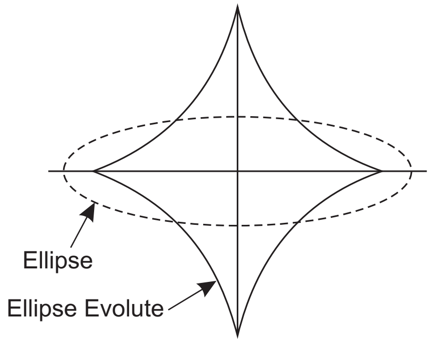


Fig. 1.15. Involute of an Ellipse

The following equations are used for defining the given:

- Circle Involute
- Catenary Involute
- Deltoid Involute
- Parabola Involute
- Parabola Involute

Circle Involute: $x = r(\cos t + t \sin t)$, $y = r(\sin t - t \cos t)$, where, r = radius of the circle, t = parameter of angle in radian.

Catenary Involute: $x = t - \tanh t$, $y = \text{sech } t$, where t be the parameter.

Deltoid Involute: $x = 2r \cos t + r \cos 2t$, $y = 2r \sin t - r \sin 2t$

where, r = radius of rolling circle involved in formation of deltoid.

Parabola Involute: $x^3 = ay^2$.

1.1.7 Envelope

A curve which touches each member of a given family of curves is called envelope of that family.

Procedure to find envelope for the given family of curves:

CASE 1: Envelope of one parameter family of curves.

Let us consider $y = f(x, \alpha)$ to be the given family of curves with 'α' as the parameter.

Step 1: Differentiate w.r.t to the parameter α partially, and find the value of the parameter.

Step 2: By substituting the value of parameter α in the given family of curves, we get the required envelope.

SPECIAL CASE: If the given equation of curve is quadratic in terms of parameter, *i.e.* $A\alpha^2 + B\alpha + c = 0$, then the envelope is given by **discriminant = 0** *i.e.* $B^2 - 4AC = 0$.

CASE 2: Envelope of two parameter family of curves.

Let us consider $y = f(x, \alpha, \beta)$ to the given family of curves, and a relation connecting the two parameters α and β, $g(\alpha, \beta) = 0$.

Step 1: Consider α as independent variable and β depends on α. Differentiate $y = f(x, \alpha, \beta) = 0$ and $g(\alpha, \beta) = 0$ w.r.t. the parameter α partially.

Step 2: Eliminating the parameters α, β from the equations resulting from step 1 and $g(\alpha, \beta) = 0$, we get the required envelope.

SOME SOLVED EXAMPLES

Example 1.11. *Show that the centre of curvature and equation of circle of curvature at the point* $\left(\frac{a}{4}, \frac{a}{4}\right)$ *on the curve* $\sqrt{x} + \sqrt{y} = \sqrt{a}$ *is* $\left(\frac{3}{4}a, \frac{3}{4}a\right)$ *and* $\left(x - \frac{3}{4}a\right)^2 + \left(y - \frac{3}{4}a\right)^2 = \frac{a^2}{2}$.

Solution. Here, the equation of the curve is

$$\sqrt{x} + \sqrt{y} = \sqrt{a} \qquad ...(i)$$

Differentiating (*i*) w.r.t. '*x*', we get

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\,y_1 = 0 \qquad ...(ii)$$

Differentiating (*ii*) w.r.t. '*x*', we get

$$-\frac{1}{4}x^{-3/2} - \frac{1}{4}y^{-3/2}\,.\,y_1\,.\,y_1 + \frac{1}{2}y^{-1/2}\,.\,y_2 = 0 \qquad ...(iii)$$

From (*ii*), at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$, we have

$$\frac{1}{2}\cdot\frac{2}{\sqrt{a}} + \frac{1}{2}\cdot\frac{2}{\sqrt{a}}\,y_1 = 0$$

$$\Rightarrow \qquad y_1 = -1$$

From equation (*iii*), at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$

$$-\frac{1}{4}\cdot\frac{4}{a}\cdot\frac{2}{\sqrt{a}} - \frac{1}{4}\cdot\frac{4}{a}\cdot\frac{2}{\sqrt{a}}(-1)^2 + \frac{1}{2}\cdot\frac{2}{\sqrt{a}}\,y_2 = 0$$

$$\Rightarrow \qquad -\frac{4}{a\sqrt{a}} + \frac{1}{\sqrt{a}}\,y_2 = 0$$

$$\Rightarrow \qquad y_2 = \frac{4}{a}$$

Now, ρ (at the given point) $= \dfrac{(1+y_1^2)^{3/2}}{y_2} = \dfrac{(1+1)^{3/2}}{4/a}$

$$= 2\sqrt{2}\,\frac{a}{4} = \frac{a}{\sqrt{2}}$$

Let $(\bar{x}, \bar{y})$ be the centre at curvature at $\left(\dfrac{a}{4}, \dfrac{a}{4}\right)$, then

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= \frac{a}{4} - \frac{-(1+1)}{4/a}$$

$$= \frac{a}{4} + \frac{a}{2} = \frac{3}{4}a$$

Similarly, $\bar{y} = y + \dfrac{1+y_1^2}{y_2}$

$$= \frac{a}{4} + \frac{1+1}{4/a}$$

$$= \frac{a}{4} + \frac{a}{2} = \frac{3}{4}a$$

∴ Equation of the circle at curvature is,

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

i.e. $\left(x - \dfrac{3}{4}a\right)^2 + \left(y - \dfrac{3}{4}a\right)^2 = \dfrac{1}{2}a^2$. **(Proved)**

Example 1.12. *Show that the equation of the evolute of the tractrix* $x = c\cos t + c\log\tan\dfrac{t}{2}$, $y = c\sin t$ *is the catenary* $y = c\cosh\dfrac{x}{c}$.

Solution. Given equation of curve is

$$x = c\cos t + c\log\tan\frac{t}{2},\ y = c\sin t$$

Differentiating w.r.t. 't', we have

$$\frac{dx}{dt} = -c\sin t + \frac{c}{\tan\frac{t}{2}}\cdot\frac{1}{2}\sec^2\frac{t}{2}$$

$$= -c\sin t + \frac{c\cos\frac{t}{2}}{\sin\frac{t}{2}}\cdot\frac{1}{2\cos^2\frac{t}{2}}$$

$$= -c\sin t + \frac{c}{2\sin\frac{t}{2}\cos\frac{t}{2}}$$

$$= -c\sin t + \frac{c}{\sin t}$$

$$= \frac{c(1-\sin^2 t)}{\sin t} = \frac{c\cos^2 t}{\sin t}$$

and $$\frac{dy}{dt} = c\cos t$$

Thus, $$y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= c\cos t \,.\, \frac{\sin t}{c\cos^2 t} = \tan t$$

and $$y_2 = \frac{d^2y}{dx^2} = \sec^2 t \,.\, \frac{dt}{dx}$$

$$= \frac{1}{\cos^2 t} \,.\, \frac{\sin t}{c\cos^2 t}$$

$$= \frac{\sin t}{c\cos^4 t}$$

Let $(\bar{x}, \bar{y})$ is the centre of curvature at any point on the curve, then

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= c\cos t + c\log\tan\frac{t}{2} - c\frac{\cos^4 t}{\sin t} \,.\, \frac{\sin t}{\cos t} \,.\, \frac{1}{\cos^2 t}$$

$$= c\cos t + c\log\tan\frac{t}{2} - c\cos t$$

or $$\bar{x} = c\log\left(\tan\frac{t}{2}\right) \qquad ...(i)$$

and $$\bar{y} = y + \frac{1+y_1^2}{y_2}$$

$$= c\sin t + \frac{1+\tan^2 t}{\dfrac{\sin t}{c\cos^4 t}}$$

$$= c\sin t + \frac{c\cos^4 t}{\sin t} \,.\, \sec^2 t$$

$$= c\sin t + \frac{c\cos^2 t}{\sin t}$$

$$= \frac{c(\sin^2 t + \cos^2 t)}{\sin t}$$

$$\bar{y} = \frac{c}{\sin t} \quad \{\because \quad \sin^2 t + \cos^2 t = 1\} \qquad ...(ii)$$

Evolute of the given curve is the locus of $(\bar{x}, \bar{y})$. Let us eliminate 't' between (i) and (ii).

From (i), $$\log \tan \frac{t}{2} = \frac{\bar{x}}{c}$$

$$\Rightarrow \quad \tan \frac{t}{2} = e^{\bar{x}/c}$$

From (ii), $$\frac{\bar{y}}{c} = \frac{1}{\sin t} = \frac{1+\tan^2 t/2}{2\tan t/2}$$

$$\frac{\bar{y}}{c} = \frac{1}{2}\left(\frac{1}{\tan t/2} + \tan\frac{t}{2}\right)$$

$$= \frac{1}{2}\left(e^{-\frac{\bar{x}}{c}} + e^{\frac{\bar{x}}{c}}\right)$$

$$\Rightarrow \quad \bar{y} = c\cos h\frac{\bar{x}}{c}$$

Changing $\bar{x}$ to x, $\bar{y}$ to y, the locus at $(\bar{x}, \bar{y})$ is

$$y = c\cos h\frac{x}{c},$$

which is the equation of evolute.

Example 1.13. *Find the coordinates of centre of curvature at any point for the given curve $x^{2/3} + y^{2/3} = a^{2/3}$. Also find the values of radius of curvature (ρ) and equation of evolute.*

Solution. The parametric equation of given curve is

$$x = a\cos^3 t, \qquad y = a\sin^3 t \qquad ...(1)$$

Differentiating w.r.t. 't', the above equation gives,

$$\frac{dx}{dt} = -3a\cos^2 t\sin t, \quad \frac{dy}{dt} = 3a\sin^2 t\cos t$$

$$\therefore \quad y_1 = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t$$

$$\therefore \quad y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}(-\tan t)$$

$$= -\sec^2 t\frac{dt}{dx}$$

$$= \frac{-\sec^2 t}{-3a\cos^2 t\sin t} = \frac{1}{3a\sin t\cos^4 t}$$

$$\therefore \quad \text{Radius of curvature } (\rho) = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= (1+\tan^2 t)^{3/2} \times 3a\sin t\cos^4 t$$

$$= \frac{3a \sin t \cos^4 t}{(\cos^2 t)^{3/2}}$$

$$= 3a \sin t \cos t$$

Now, the centre of curvature $(\bar{x}, \bar{y})$ at the point 't', is given by

We have, $\bar{x} = x - \rho \sin \psi$

$$= x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= x + \tan t\,(1 + \tan^2 t) \times 3a \sin t \cos^4 t$$

$\Rightarrow$ $\bar{x} = x + 3a \sin^2 t \cos t$

$$= a \cos^3 t + 3a \sin^2 t \cos t \qquad ...(2)$$

$[\because \quad x = a \cos^3 t]$

Similarly, $\bar{y} = y + \dfrac{(1+y_1^2)}{y_2}$

$$\bar{y} = y + \frac{(1+\tan^2 t)}{\dfrac{1}{3a \sin t \cos^4 t}}$$

$$= y + (1 + \tan^2 t)\,(3a \sin t \cos^4 t)$$

$$= y + 3a \sin t \cos^2 t$$

$$= a \sin^3 t + 3a \sin t \cos^2 t \qquad ...(3)$$

Eliminating x, y and t from equation (1), (2) and (3), we get

$$\bar{x} + \bar{y} = a\,(\cos^3 t + \sin^3 t + 3 \sin^2 t \cos t + 3 \cos^2 t \sin t)$$

$$= a(\cos t + \sin t)^3$$

or $(\bar{x} + \bar{y})^{2/3} = a^{2/3} (\cos t + \sin t)^2 \qquad ...(4)$

Similarly, $(\bar{x} - \bar{y})^{2/3} = a^{2/3} (\cos t - \sin t)^2 \qquad ...(5)$

Adding (4) and (5), we have

$$(\bar{x} + \bar{y})^{2/3} + (\bar{x} - \bar{y})^{2/3} = a^{2/3} (\cos^2 t + \sin^2 t + 2 \cos t \sin t + \cos^2 t + \sin^2 t - 2 \cos t \sin t)$$

$$= a^{2/3}(2) = 2a^{2/3}$$

So, the locus of $(\bar{x}, \bar{y})$ *i.e.*, the equation of evolute is

$$(x + y)^{2/3} + (x - y)^{2/3} = 2a^{2/3}$$

and the coordinates of the centre of curvature are $[(a \cos^3 t + 3 \sin^2 t \cos t), (a \sin^3 t + 3 \sin t \cos^2 t)]$.

Example 1.14. *For the given rectangular hyperbola $xy = a^2$ (i.e. $x = at$, $y = a/t$)*

i. *find the radius of curvature (ρ)*

ii. *find the coordinates of centre of curvature (i.e. $\bar{x}, \bar{y}$)*

iii. *shows that the evolute of given curve is $(x + y)^{2/3} - (x - y)^{2/3} = (4a)^{2/3}$.*

Solution. i. To find radius of curvature, we have

$$xy = a^2, \qquad \text{or} \quad y = \frac{a^2}{x}$$

Therefore, $y_1 = \frac{dy}{dx} = \frac{-a^2}{x^2}$ and $y_2 = \frac{d^2y}{dx^2} = \frac{2a^2}{x^3}$

Thus,
$$\rho = \frac{(1+{y_1}^2)^{3/2}}{y_2}$$
$$= \frac{\left(1+\frac{a^4}{x^4}\right)^{3/2}}{\frac{2a^2}{x^3}}$$

$\Rightarrow$
$$\rho = \frac{x^3}{2a^2}\left(\frac{x^4+a^4}{x^4}\right)^{3/2}$$

ii. To find coordinates of centre of curvatures, we have

$$\bar{x} = x - \frac{y_1(1+{y_1}^2)}{y_2} \quad \text{and} \quad \bar{y} = y + \frac{1+{y_1}^2}{y_2}$$

$\Rightarrow$
$$\bar{x} = x - \frac{\frac{-a^2}{x^2}\left(1+\frac{a^4}{x^4}\right)}{\frac{2a^2}{x^3}}$$
$$= x + \frac{x^4+a^4}{2x^3}$$
$$= \frac{2x^4+x^4+a^4}{2x^3}$$
$$= \frac{3x^4+a^4}{2x^3} \qquad \text{[Since } xy = a^2\text{]}$$

We have
$$\frac{3x^4+x^2y^2}{2x^3} = \frac{3}{2}x + \frac{y^2}{2x}$$

Similarly,
$$\bar{y} = y + \frac{1+{y_1}^2}{y_2}$$
$$= y + \frac{1+\frac{a^4}{x^4}}{\frac{2a^2}{x^3}}$$
$$= y + \frac{x^4+a^4}{2a^2x}$$
$$= y + \frac{x^4+x^2y^2}{2x^2y} \qquad \text{[Since } xy = a^2\text{]}$$

$$= y + \frac{x^2 + y^2}{2y} = \frac{x^2 + 3y^2}{2y} = \frac{3}{2}y + \frac{x^2}{2y}$$

Thus, the coordinates of the centre of curvature are

$$(\bar{x}, \bar{y}) = \left(\frac{3x}{2} + \frac{y^2}{2x}, \frac{3y}{2} + \frac{x^2}{2y} \right)$$

iii. Further, to show the equation of the evolute of the given curve is $(x + y)^{2/3} - (x - y)^{2/3} = (4a)^{2/3}$, we have

$$(\bar{x} + \bar{y}) = \frac{1}{2xy}[x^3 + y^3 + 3x^2y + 3xy^2]$$

$$= \frac{1}{2xy}(x + y)^3 = \frac{1}{2a^2}(x + y)^3$$

and do, $$(\bar{x} + \bar{y})^{2/3} = \frac{1}{(2a^2)^{2/3}}(x + y)^2 \qquad \{\because \quad xy = a^2]$$

Similarly, $$(\bar{x} - \bar{y})^{2/3} = \frac{1}{(2a^2)^{2/3}}(x - y)^2$$

Therefore, we have

$$(\bar{x} + \bar{y})^{2/3} - (\bar{x} - \bar{y})^{2/3} = \frac{1}{(2a^2)^{2/3}}[(x + y)^2 - (x - y)^2]$$

$$= \frac{1}{(2a^2)^{2/3}}(4xy) = \frac{1}{(2a^2)^{2/3}}(4a^2) = (4a)^{2/3}$$

Thus, the locus of $(\bar{x}, \bar{y})$ is

$$(\bar{x} + \bar{y})^{2/3} - (\bar{x} - \bar{y})^{2/3} = (4a)^{2/3}.$$

Example 1.15. *Find the centre of curvature of the following curves:*

i. $y = x^3 - 6x^2 + 3x + 1$ *at* $(1, -1)$

ii. *the parabola* $y^2 = 4ax$ *at* (x, y). *Also find the equation of the evolute of the given curve.*

iii. *the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *at* (x, y). *Also find its evolute.*

Solution. i. The given curve is

$$y = x^3 - 6x^2 + 3x + 1$$

Therefore, $$y_1 = \frac{dy}{dx} = 3x^2 - 12x + 3$$

so $$y_1 \text{ at } (1, -1) = -6$$

and $$y_2 = \frac{d^2y}{dx^2} = 6x - 12$$

$\therefore$ $$y_2 \text{ at } (1, -1) = -6$$

Thus, $$\bar{x} = x - \frac{y_1(1 + y_1^2)}{y_2}$$

$$= 1 - \frac{(-6)[1+(-6)^2]}{-6} = -36$$

Similarly, $$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= -1 + \frac{1+(-6)^2}{-6} = -\frac{43}{6}$$

Hence the centre of curvature are $(\bar{x}, \bar{y}) = (-36, -43/6)$.

ii. The given equation of the parabola is $y^2 = 4ax$.

Therefore, $$y = 2\sqrt{ax} \quad \Rightarrow \quad y_1 = 2\sqrt{a}\,.\frac{1}{2\sqrt{x}} = \sqrt{\frac{a}{x}}$$

and $$y_2 = \frac{d^2y}{dx^2} = -\frac{1}{2}\sqrt{a}\,x^{-3/2}$$

Therefore, $$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= x - \frac{\sqrt{\frac{a}{x}}\left(1+\frac{a}{x}\right)}{-\frac{1}{2}\sqrt{a}x^{-3/2}} = x + 2(x+a)$$

$$= 3x + 2a \qquad ...(1)$$

and $$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= y + \frac{\left(1+\frac{a}{x}\right)}{-\frac{1}{2}\sqrt{a}x^{-3/2}}$$

$$= 2\sqrt{a}\sqrt{x} - \frac{2(x+a)}{\sqrt{a}x^{-1/2}} \qquad \{\because \quad y = 2\sqrt{ax}\ \}$$

$$= 2\sqrt{a}\sqrt{x}\left(1 - \frac{x-a}{a}\right)$$

$$= -2\,\frac{x^{3/2}}{\sqrt{a}} \qquad ...(2)$$

Hence the centre of curvature of the given curve is

$$(\bar{x}, \bar{y}) = \left(3x + 2a, -\frac{2x^{3/2}}{\sqrt{a}}\right)$$

From (1), we have $$x = \frac{\bar{x} - 2a}{3}$$

Putting this value in (2), we have

$$\bar{y} = -\frac{2\left(\frac{\bar{x}-2a}{3}\right)^{3/2}}{\sqrt{a}}$$

or $$a\bar{y}^2 = 4\left(\frac{\bar{x}-2a}{3}\right)^3$$

or $$27\,a\bar{y}^2 = 4(\bar{x}-2a)^3$$

Therefore, the locus of the centre of curvature $(\bar{x}, \bar{y})$ is $27ay^2 = 4(x - 2a)^3$, which is the required evolute.

iii. The given equation of the ellipse is,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1,$$

Therefore, $y_1 = \frac{dy}{dx} = \frac{-b^2x}{a^2y}$ and $y_2 = \frac{d^2y}{dx^2} = \frac{-b^4}{a^2y^3}$

Therefore, $(\bar{x}, \bar{y})$ centre of curvature are,

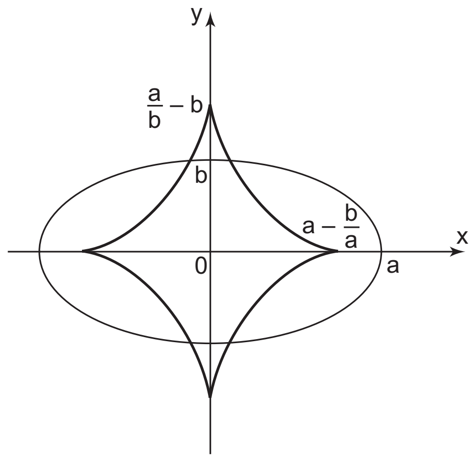


Fig. 1.16

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= x - \frac{-\frac{b^2x}{a^2y}\left(1+\frac{b^4x^2}{a^4y^2}\right)}{-\frac{b^4}{a^2y^3}}$$

$$= x - \frac{x}{a^4b^2}(a^4y^2+b^4x^2)$$

$$= x - \frac{x}{a^4b^2}\,[a^2b^2(a^2-x^2)+b^4x^2]$$

$$= \frac{a^2-b^2}{a^4}x^3 \qquad \text{...(1)}$$

Similarly, $$\bar{y} = y + \left(\frac{1+y_1^2}{y_2}\right)$$

$$= y + \frac{1+\frac{b^4x^2}{a^4y^2}}{-\frac{b^4}{a^2y^3}}$$

$$= y - \frac{y}{a^2b^4}\,(a^4y^2+b^4x^2)$$

$$= y - \frac{y}{a^2b^4}\,[a^4y^2+b^2a^2(b^2-y^2)]$$

$$= \frac{b^2 - a^2}{b^4} y^3 \qquad \ldots(2)$$

From eqn. (1), we have

$$x = \left(\frac{a^4 \bar{x}}{a^2 - b^2}\right)^{1/3}$$

and from eqn. (2), we have

$$y = \left(\frac{b^4 \bar{y}}{b^2 - a^2}\right)^{1/3}$$

Substituting these values in equation of ellipse, we have

$$\frac{1}{a^2}\left(\frac{a^4 \bar{x}}{a^2 - b^2}\right)^{2/3} + \frac{1}{b^2}\left(\frac{b^4 \bar{y}}{b^2 - a^2}\right)^{2/3} = 1$$

or $\quad (a\bar{x})^{2/3} + (b\bar{y})^{2/3} = (a^2 - b^2)^{2/3}$

Therefore the locus of the centre of curvature $(\bar{x}, \bar{y})$ is

$$(a\bar{x})^{2/3} + (b\bar{y})^{2/3} = (a^2 - b^2)^{2/3}$$

which is the required evolute for the given ellipse.

Example 1.16. *Find the envelope of the family of straight line* $y = mx + \sqrt{a^2m^2 + b^2}$ *, m is the parameter.*

Solution. Given equation of family of curves

$$y = mx + \sqrt{a^2m^2 + b^2}$$

$$\Rightarrow \quad (y - mx) = \sqrt{a^2m^2 + b^2}$$

$$\Rightarrow \quad (y - mx)^2 = (a^2m^2 + b^2)$$

$$\Rightarrow \quad y^2 + m^2x^2 - 2mxy = a^2m^2 + b^2$$

$$\Rightarrow \quad m^2(x^2 - a^2) - 2mxy + (y^2 - b^2) = 0$$

Differentiate partially w.r.t the parameter (*i.e.* m)

$$\Rightarrow \quad 2m(x^2 - a^2) - 2xy = 0$$

$$\Rightarrow \quad m = \frac{xy}{(x^2 - a^2)}$$

Substituting the value of m in the given family of curves

$$\Rightarrow \quad m^2(x^2 - a^2) - 2mxy + (y^2 - b^2) = 0$$

$$\Rightarrow \quad \left(\frac{xy}{(x^2 - a^2)}\right)^2 (x^2 - a^2) - 2\frac{xy}{(x^2 - a^2)} xy + (y^2 - b^2) = 0$$

$$\Rightarrow \quad \frac{x^2y^2}{x^2 - a^2} - \frac{2x^2y^2}{x^2 - a^2} + (y^2 - b^2) = 0$$

$$\Rightarrow \quad -\frac{x^2y^2}{x^2 - a^2} + (y^2 - b^2) = 0$$

$$\Rightarrow \quad \frac{x^2y^2}{x^2 - a^2} = (y^2 - b^2)$$

$$\Rightarrow \quad x^2y^2 = (x^2 - a^2)(y^2 - b^2)$$

$$\Rightarrow \quad x^2y^2 = x^2y^2 - x^2b^2 - a^2y^2 + a^2b^2$$

$$\Rightarrow \quad x^2b^2 + a^2y^2 = a^2b^2$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\therefore$ The envelope of the given family of straight lines is an ellipse.

Example 1.17. *Find the envelope of $y = mx + am^p$ where m is the parameter and a, p are constants.*

Solution. Given, $y = mx + am^p$ ...(1)

Differentiate the equation (1) w.r.t. parameter m, we get

$$0 = x + pam^{p-1}$$

$$\Rightarrow \quad m = \left(\frac{-x}{pa}\right)^{\frac{1}{p-1}} \qquad ...(2)$$

Using (2) and eliminates m from (1)

$$y = \left(\frac{-x}{pa}\right)^{\frac{1}{p-1}} x + a\left(\frac{-x}{pa}\right)^{\frac{p}{p-1}}$$

$$\Rightarrow \quad y^{p-1} = \left(\frac{-x}{pa}\right) x^{p-1} + a^{p-1}\left(\frac{-x}{pa}\right)^p$$

i.e. $\quad apy^{p-1} = (-x)^p + a^{p-2}(-x)^p$

which is the required equation of envelope of (1).

Problems Based on Envelope of Two Parameter Family of Curves:

Example 1.18. *Find the envelope of family of straight lines $ax + by = 1$, where a and b are parameters connected by the relation $ab = 1$.*

Solution. Given, $ax + by = 1$ ...(1)

and $ab = 1$ ...(2)

Differentiating (1) w.r.t 'a' (considering 'a' as independent variable and 'b' depends on a).

$$x + \frac{db}{da}y = 0$$

i.e. $$\frac{db}{da} = \frac{-x}{y} \qquad ...(3)$$

Differentiating (2) w.r.t 'a'

$$b + a\frac{db}{da} = 0$$

i.e. $$\frac{db}{da} = \frac{-b}{a} \qquad ...(4)$$

From (3) and (4), we have

$$\frac{x}{y} = \frac{b}{a}$$

i.e. $$\frac{ax}{1} = \frac{by}{1} = \frac{ax+by}{2} = \frac{1}{2}$$

∴ $$a = \frac{1}{2x} \quad \text{and} \quad b = \frac{1}{2y} \qquad ...(5)$$

Using (5) in (2), we get the envelope as $4xy = 1$.

Example 1.19. *Find the envelope of family of curve* $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$, *where a and b are parameters connected by the relation* $\sqrt{a} + \sqrt{b} = 1$.

Solution. Given, $$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1 \qquad ...(1)$$

and $$\sqrt{a} + \sqrt{b} = 1 \qquad ...(2)$$

Differentiating (1) with respect to 'a'

$$\frac{\sqrt{x}}{-2a^{3/2}} + \frac{\sqrt{y}}{-2b^{3/2}} \frac{db}{da} = 0$$

i.e. $$\frac{db}{da} = \frac{-\sqrt{x}\, b^{3/2}}{\sqrt{y}\, a^{3/2}} \qquad ...(3)$$

Differentiating (2) with respect to 'a'

$$\frac{1}{2\sqrt{a}} + \frac{1}{2\sqrt{b}} \frac{db}{da} = 0$$

i.e. $$\frac{db}{da} = \frac{-\sqrt{b}}{\sqrt{a}} \qquad ...(4)$$

From (3) and (4), we have

$$\frac{\sqrt{x}}{\sqrt{y}} \frac{b}{a} = 1$$

i.e. $$\frac{\sqrt{\frac{x}{a}}}{\sqrt{a}} = \frac{\sqrt{\frac{y}{b}}}{\sqrt{b}} = \frac{\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}}}{\sqrt{a} + \sqrt{b}} = \frac{1}{1}$$

∴ $$a = \sqrt{x} \text{ and } b = \sqrt{y} \qquad ...(5)$$

Using (5) in (2), we get the envelope as $x^{1/4} + y^{1/4} = 1$.

Example 1.20. *Find the envelope of family of straight line* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, *where a, b are two parameters which are connected by the relation* $a + b = c$.

Solution. Given equation of family of straight lines is

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad ...(1)$$

Also given, $$a + b = c$$

⇒ $$b = c - a \qquad ...(2)$$

Substituting (2) in (1), we get

$$\frac{x}{a}+\frac{y}{c-a} = 1$$

Differentiate w.r.t. a partially, we get

$$-\frac{x}{a^2}+\frac{y}{(c-a)^2} = 0$$

$$\Rightarrow \quad \frac{x}{a^2} = \frac{y}{(c-a)^2}$$

$$\Rightarrow \quad \frac{(c-a)^2}{a^2} = \frac{y}{x}$$

$$\Rightarrow \quad \left(\frac{c-a}{a}\right)^2 = \frac{y}{x}$$

$$\Rightarrow \quad \frac{c-a}{a} = \sqrt{\frac{y}{x}}$$

$$\Rightarrow \quad \frac{c}{a}-1 = \sqrt{\frac{y}{x}}$$

$$\Rightarrow \quad \frac{c}{a} = 1+\sqrt{\frac{y}{x}}$$

$$\Rightarrow \quad \frac{c}{a} = \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}}$$

$$\Rightarrow \quad a = \frac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}}$$

Now, substitute the value of a in $b = c - a$

$$\Rightarrow \quad b = c - \frac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}}$$

$$= \frac{c\sqrt{x}+c\sqrt{y}-c\sqrt{x}}{\sqrt{x}+\sqrt{y}}$$

$$\Rightarrow \quad b = \frac{c\sqrt{y}}{\sqrt{x}+\sqrt{y}}$$

Now, substitute the values of a and b in the given family of curves $\frac{x}{a}+\frac{y}{b} = 1$, we get

$$\frac{x}{\left(\frac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}}\right)}+\frac{y}{\left(\frac{c\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)} = 1$$

$$\Rightarrow \quad \frac{x(\sqrt{x}+\sqrt{y})}{c\sqrt{x}}+\frac{y(\sqrt{x}+\sqrt{y})}{c\sqrt{y}} = 1$$

$$\Rightarrow \quad \frac{x(\sqrt{x}+\sqrt{y})}{\sqrt{x}} + \frac{y(\sqrt{x}+\sqrt{y})}{\sqrt{y}} = c$$

$$\Rightarrow \quad (\sqrt{x}+\sqrt{y})\left(\frac{x}{\sqrt{x}} + \frac{y}{\sqrt{y}}\right) = 0$$

$$\Rightarrow \quad (\sqrt{x}+\sqrt{y})(\sqrt{x}+\sqrt{y}) = c$$

$$\Rightarrow \quad (\sqrt{x}+\sqrt{y})^2 = c$$

$$\Rightarrow \quad (\sqrt{x}+\sqrt{y}) = \sqrt{c}$$ is the required envelope.

Problems Based on Evolute as Envelope of its Normals:

Example 1.21. *By considering the evolute of a curve as the envelope of its normal, find the evolute of*

$$x = \cos\theta + \theta\sin\theta,\ y = \sin\theta - \theta\cos\theta.$$

Solution. Given $x = \cos\theta + \theta\sin\theta$, $y = \sin\theta - \theta\cos\theta$, then

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\theta\sin\theta}{\theta\cos\theta} = \tan\theta$$

Equation of normal line to the hyperbola is

$$(y - (\sin\theta - \theta\cos\theta)) = \frac{-1}{\tan\theta}(x - (\cos\theta + \theta\sin\theta))$$

$$\Rightarrow \quad y\sin\theta - \sin^2\theta + \theta\sin\theta\cos\theta = -x\cos\theta + \cos^2\theta + \theta\sin\theta\cos\theta$$

i.e. $\quad y\sin\theta + x\cos\theta = 1 \quad$ ...(1)

Differentiating (1) with respect to the parameter θ, we have

$$y\cos\theta - x\sin\theta = 0 \quad ...(2)$$

Multiplying (1) by cos θ and (2) by sin θ and then subtracting, we have

$$x = \cos\theta \quad ...(3)$$

Similarly, we get, $\quad y = \sin\theta \quad$ ...(4)

Eliminating θ between (3) and (4) we get the required evolute as $x^2 + y^2 = 1$.

Example 1.22. *Determine the evolute of hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *by considering it as an envelope of its normal.*

Solution. Try yourself.

EXERCISE 1.2

1. Find the radius of curvatures and the centre of curvatures for the curve $y = \tan x$ at the point where $x = \pi/4$.
2. Find the centre of the curvatures for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, and also obtain its evolute.

3. Show that the evolute of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ is another equal cycloid.
4. Show that the evolute of the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ is $x^2 + y^2 = a^2$.
5. Determine the envelope of $x\sin\theta - y\cos\theta = a\theta$, where θ being the parameter.
6. Find the envelope of $x\sec^2\theta + y\,\text{cosec}^2\theta = a$, where θ is the parameter.
7. Find the envelope of family of straight lines $\frac{x}{a} + \frac{y}{b} = 1$, where a, b are parameters, connected by the relation $a^2b^3 = c^5$.
8. Find the envelope of the family of circles whose centres lie on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and which passes through its centre.
9. Determine the equation of the envelope of family of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where the parameters '$a$' and '$b$' are connected by the relation $\frac{a^2}{l^2} + \frac{b^2}{m^2} = 1$, l and m are non-zero constants.

Answers

1. $\frac{5\sqrt{5}}{4}, \left(\frac{\pi - 10}{4}, \frac{9}{4}\right)$
2. **Hint:** $x = a\sec\theta$, $y = b\tan\theta$ centre of curvature $\left(\frac{a^2 + b^2}{a\cos^3\theta}, \frac{-\sin^3\theta\,(a^2 + b^2)}{\cos^3\theta}\right)$

   evolute $(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}$.

5. $x = a(\sin\theta + \cos\theta)$, $y = a(\sin\theta - \theta\cos\theta)$
6. $x^2 + y^2 - 2xy - 2ax - 2ay + a^2 = 0$
7. $x^2y^3 = \frac{72}{3125}c^5$
8. $(x^2 + y^2)^2 = 4(a^2x^2 + b^2y^2)$
9. $\frac{x}{l} + \frac{y}{m} = 1$

INTERESTING FACT

Have you ever observed the machines or the toys that contains a winding key, such as those instrument playing monkeys? The inside spiral spring undergoes a motion in a "***circular involute***".

VIDEO REFERENCES

Curvature and Evolutes

USES OF ICT

http://kmr.csc.kth.se/wp/research/math-rehab/learning-object-repository/geometry-2/metric-geometry/euclidean-geometry/geometry/plane-curves/evolutes/

APPLICATIONS TO REAL LIFE

- They are used in mechanical industries, especially the teeth industries, where teeth of revolving machines and gears are made, to minimize the vibrations as much as possible.
- Scroll and Gas compressors are two such machines used to pump, compress or pressurize fluids. There shape is an application of this concept, which makes sure that they are efficient and less noisy.
- The road safety needs to be kept in mind while designing road curvatures, and similarly the size of grinding wheel also needs to be considered. The concept of curvature comes into play at that time.

1.2 EVALUATION OF DEFINITE AND IMPROPER INTEGRAL

1.2.1 Definite Integral

A definite integral is denoted by $\int_a^b f(x)\,dx$ where 'a' is called the lower limit of the integral and 'b' is called the upper limit of the integral. The definite integral is introduced either as the limit of a sum or if it has anti-derivative F in the interval $[a, b]$, then its value is the difference between the values of F at the end points *i.e.*, $F(b) - F(a)$. The definite integral has a unique value.

1.2.1.1 Definite Integral as the Limit of a Sum

Let f be a continuous function defined on close interval $[a, b]$. Assume that all the values taken by the function are non-negative, so the graph of the function is a curve above the x-axis.

The definite integral $\int_a^b f(x)\,dx$ is the area bounded by the curve $y = f(x)$, the ordinates $x = a, x = b$ and the x-axis.

When we evaluate this area, it is equal to

$$\int_a^b f(x)\,dx = (b-a) \lim_{n\to\infty} \frac{1}{n} [f(a) + f(a+h) + \ldots + f(a+(n-1)\,h)]$$

where $$h = \frac{b-a}{n} \to 0 \text{ as } n \to \infty$$

The above expression is known as the definition of definite integral as the limit of sum.

SOME SOLVED EXAMPLES

Example 1.23. *Find* $\int_0^2 (x^2+1)\,dx$ *as the limit of a sum.*

Solution. By definition

$$\int_a^b f(x)\,dx = (b-a) \lim_{n\to\infty} \frac{1}{n} [f(a) + f(a+h) + \ldots + f(a+(n-1)h)],$$

where $$h = \frac{b-a}{n}$$

Therefore,
$$\int_0^2 (x^2+1)\,dx = 2\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f\left(\frac{2}{n}\right)+f\left(\frac{4}{n}\right)+\ldots+f\left(\frac{2(n-1)}{n}\right)\right]$$
$$= 2\lim_{n\to\infty}\frac{1}{n}\left[1+\left(\frac{2^2}{n^2}+1\right)+\left(\frac{4^2}{n}+1\right)+\ldots+\left(\frac{(2n-2)^2}{n^2}+1\right)\right]$$
$$= 2\lim_{n\to\infty}\frac{1}{n}\left[\underbrace{(1+1+\ldots+1)}_{n\text{ times}}+\frac{1}{n^2}(2^2+4^2+\ldots+(2n-2)^2\right]$$
$$= 2\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{2^2}{n^2}(1^2+2^2+\ldots+(n-1)^2\right]$$
$$= 2\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{4}{n^2}\frac{n(n-1)(2n-1)}{6}\right]$$
$$= 2\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{2}{3}\frac{(n-1)(2n-1)}{n}\right]$$
$$= 2\lim_{n\to\infty}\left[1+\frac{2}{3}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\right]$$
$$= 2\left[1+\frac{4}{3}\right]=\frac{14}{3}.$$

Example 1.24. *Evaluate* $\int_0^2 e^x dx$ *as the limit of a sum.*

Solution. By definition
$$\int_0^2 e^x dx = (2-0)\lim_{n\to\infty}\frac{1}{n}\left[e^0+e^{2/n}+e^{4/n}+\ldots+e^{(2n-2)/n}\right]$$
Using the sum of n terms of a G.P., where $a = 1$, $r = e^{2/n}$,

We have,
$$\int_0^2 e^x dx = 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^{\frac{2n}{n}}-1}{e^{\frac{2}{n}}-1}\right]$$
$$= 2\lim_{n\to\infty}.\frac{1}{n}\left[\frac{e^2-1}{e^{2/n}-1}\right]$$
$$= \frac{\frac{2(e^2-1)}{1}}{\lim_{n\to\infty}\left(\frac{e^{2/n}-1}{2/n}\right).2} = e^2-1 \qquad \left[\text{using } \lim_{h\to 0}\frac{e^n-1}{h}=1\right]$$

FUNDAMENTAL THEOREM OF CALCULUS

1.2.2. First Fundamental Theorem of Integral Calculus

If $f(x)$ is defined in the interval $[a, b]$, then the definite integral of $f(x)$ is defined as

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

where $\frac{d}{dx} F(x) = f(x)$

The definite integral defined above denotes the area bounded by the curve $y = f(x)$, the x-axis and two ordinates at $x = a$ and $x = b$.

SOME SOLVED EXAMPLES

Example 1.25. *Evaluate* $\int_0^2 x^2\,dx$.

Solution. Given,

$$\int_0^2 x^2\,dx = \left|\frac{x^3}{3}\right|_0^2 = \left[\frac{(2)^3}{3} - 0\right] = \frac{8}{3}.$$

Example 1.26. *Evaluate* $\int_0^{2\pi} \cos x\,dx$.

Solution.

$$\int_0^{2\pi} \cos x\,dx = \left| -\sin x \right|_0^{2\pi} = [-\sin 2\pi + \sin 0] = [-0 + 0] = 0$$

Example 1.27. *Evaluate* $\int_2^3 (x+1)\,dx$.

Solution. Given,

$$\int_2^3 (x+1)\,dx = \left[\frac{x^2}{2} + x\right]_2^3$$

$$= \left[\left(\frac{(3)^2}{2} + 3\right) - \left(\frac{(2)^2}{2} + 2\right)\right]$$

$$= \left[\left(\frac{9}{2} + 3\right) - \left(\frac{4}{2} + 2\right)\right]$$

$$= \left[\left(\frac{9+6}{2}\right) - \left(\frac{4+4}{2}\right)\right]$$

$$= \left[\frac{15}{2} - \frac{8}{2}\right]$$

$$= \frac{7}{2}.$$

1.2.3 Second Fundamental Theorem of Integral Calculus

Second fundamental theorem of calculus states that if $f(x)$ is continuous in the interval $[a, b]$ and F is the indefinite integral of $f(x)$ on $[a, b]$, then

$$F'(x) = f(x)$$

Mathematically, if $\quad F(x) = \int_a^x f(t)\,dt$

then, $\quad F'(x) = f(x)$

> **Remark:** As anti-derivatives and derivatives are opposites to each other, if you derive the anti derivative of the function, you will get original function.

Example 1.28. *Solve the given with the help of 2nd Fundamental theorem of Integral Calculus*

$$F(x) = \int_0^{x^3} (t^2 + t)\,dt$$

Solution. Given $\quad F(x) = \int_0^{x^3} (t^2 + t)\,dt$

$$= \left[\frac{t^3}{3} + \frac{t^2}{2}\right]_0^{x^3} = F(x^3) - F(0)$$

$$F(x) = \left(\frac{(x^3)^3}{3} - \frac{(x^3)^2}{2}\right) - \left(\frac{(0)^3}{3} - \frac{(0)^2}{2}\right)$$

$$F(x) = \frac{x^9}{3} - \frac{x^6}{2}$$

$$F'(x) = 3x^8 - 3x^5$$

$$F'(x) = 3x^2\,((x^3)^2 + (x^3))$$

$3x^2$ is the derivative of the upper limit x^3 and $((x^3)^2 + (x^3))$ is the same as $(t^2 + t)$.

By the end of this equation, we can see that the derivative of $F(x)$, which is the integral of $f(x)$, is equivalent to the original function $f(x)$. The functions of $F'(x)$ and $f(x)$ are extremely similar.

Example 1.29. *Solve the given* $F(x) = \int_0^{x^2} (t+7)^{1/2}\,dt$ *with the help of 2nd Fundamental theorem of Integral Calculus.*

Solution. Given $\quad F(x) = \int_0^{x^2} (t+7)^{1/2}\,dt$

$$= \left[\frac{2(t+7)^{3/2}}{3}\right]_0^{x^2}$$

$$= F(x^2) - F(0)$$

$$F(x) = \left(\frac{2(x^2+7)^{3/2}}{3}\right) - \left(\frac{2(0+7)}{3}\right)^{3/2}$$

$$F(x) = \frac{2(x^2+7)^{3/2}}{3} - \frac{2(7)^{3/2}}{3}$$

$$F'(x) = 2x\,(x^2 + 7)^{1/2}$$

$2x$ is the derivative of the upper limit x^2 and $(x^2 + 7)^{1/2}$ is same as $(t + 7)^{1/2}$.

Example 1.30. *Solve the given* $F(x) = \int_{-3}^{\sqrt{x}} (3t^2 - 30)\,dt$ *with the help of 2nd Fundamental theorem of Integral Calculus.*

Solution. Given
$$F(x) = \int_{-3}^{\sqrt{x}} (3t^2 - 30)\,dt$$
$$= \left[t^3 - 30t\right]_{-3}^{\sqrt{x}} = F(\sqrt{x}) - F(-3)$$
$$F(x) = [(\sqrt{x})^3 - 30(\sqrt{x})] - [(-3)^3 - 30(-3)]$$
$$F'(x) = \frac{3}{2}x^{1/2} - \frac{15}{x^{1/2}}$$
$$F'(x) = \frac{1}{2\sqrt{x}}\,(3x - 30)$$

$\frac{1}{2\sqrt{x}}$ is the derivative of the upper limit $\sqrt{x}$ and $(3x - 30)$ is same as $(3t^2 - 30)$.

1.2.4 Properties of Definite Integrals

Here we define some properties of definite integrals which are very useful in evaluating them.

1. If $f_1(x)$ and $f_2(x)$ are continuous and bounded functions over the interval $[a, b]$ and k_1 and k_2 are two constants, then
$$\int_a^b [k_1 f_1(x) + k_2 f_2(x)]\,dx = k_1 \int_a^b f_1(x)\,dx + k_2 \int_a^b f_2(x)\,dx$$
This is called linearity property.

2. $$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$
Both sides are equal to $F(b) - F(a)$, it shows that variable in integration is dummy.

3. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$
For $F(b) - F(a) = -\{F(a) - F(b)\}$

4. $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$
Here, 'c' is defined as $a < c < b$.
R.H.S. is equal to $F(c) - F(a) + F(b) - F(c)$ which is equal to $F(b) - F(a)$.

5. $$\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx$$
This is known as invariance property and can be prove as on putting $a - x = t$, we have $-dx = dt$.
$$\int_0^a f(a - x)\,dx = -\int_a^0 f(t)\,dt = \int_0^a f(t)\,dt \qquad \text{[By Property (3)]}$$
$$= \int_0^a f(x)\,dx \qquad \text{[By Property (2)]}$$

6. $$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \quad \text{if } f(x) \text{ is even } i.e.\ f(-x) = f(x)$$
$$= 0 \quad \text{if } f(x) \text{ is odd } i.e.\ f(-x) = -f(x)$$

Proof. $$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \qquad ...(1)$$

Putting $x = -t$ in the first integral on the right, we have

$$\int_{-a}^{0} f(x)\,dx = -\int_{a}^{0} f(-t)\,dt = \int_{0}^{a} f(-t)\,dt \qquad \text{[By Prop. (3)]}$$
$$= \int_{0}^{a} f(-x)\,dx \qquad \text{[By Prop. (2)]}$$

Substituting this in (1), we have

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \{f(x) + f(-x)\}\,dx$$
$$= \begin{cases} 2\int_{0}^{a} \{f(x)\,dx, \text{if } f(-x) = f(x) \\ \quad 0, \quad \text{if } f(-x) = -f(x) \end{cases}$$

Function like x^4, cos x etc. for which $f(-x) = f(x)$ are even functions.
Functions like x^3, sin x etc. for which $f(-x) = -f(x)$ are called odd functions.

7. $$\int_{0}^{2a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \quad \text{if } f(2a - x) = f(x)$$
$$= 0, \quad \text{if } f(2a - x) = -f(x)$$

The property can be proved in a manner similar to property (6).

SOME SOLVED EXAMPLES

Example 1.31. *Evaluate* $\int_{0}^{\pi/2} \log \sin x\,dx.$

Solution. Given $$I = \int_{0}^{\pi/2} \log \sin x\,dx$$

Then, $$I = \int_{0}^{\pi/2} \log \cos x\,dx \qquad \text{[By Property (5)]}$$

Adding the two value of I, we get

$$2I = \int_{0}^{\pi/2} (\log \sin x + \log \cos x)\,dx$$
$$= \int_{0}^{\pi/2} \{\log (2 \sin x \cos x) - \log 2\}\,dx$$
$$= \int_{0}^{\pi/2} \log \sin 2x\,dx - \frac{1}{2}\pi \log 2$$
$$= \frac{1}{2}\int_{0}^{\pi} \log \sin u\,du - \frac{1}{2}\pi \log 2, \text{ where } x = \frac{u}{2}$$
$$= \int_{0}^{\pi/2} \log \sin u\,du - \frac{1}{2}\pi \log 2, \qquad \text{[By Property (7)]}$$

$$= I - \frac{1}{2}\pi \log 2$$

Therefore, $$I = -\frac{1}{2}\pi \log 2,$$

$$\therefore \quad \int_0^{\pi/2} \log \sin x\, dx = -\frac{1}{2}\pi \log 2.$$

Example 1.32. *Evaluate* $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$.

Solution. Let $$I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)} dx \qquad \text{[By Property (5)]}$$

$$= \int_0^{\pi} \frac{-(\pi - x)\tan x}{-\sec x - \tan x} dx$$

$$= \pi \int_0^{\pi} \frac{\tan x\, dx}{\sec x + \tan x} - I$$

$$\therefore \quad 2I = \pi \int_0^{\pi} \frac{\tan x}{\sec x + \tan x} dx$$

$$= \pi \int_0^{\pi} \frac{\tan x(\sec x - \tan x)}{\sec^2 x - \tan^2 x} dx$$

$$= \pi \int_0^{\pi} \frac{\sec x \tan x - \tan^2 x}{1} dx$$

$$= \pi \int_0^{\pi} (\sec x \tan x - \sec^2 x + 1)\, dx$$

$$= \pi\left[\sec x - \tan x + x\right]_0^{\pi} = \pi(\pi - 2)$$

or $$I = \frac{\pi}{2}(\pi - 2)$$

Therefore,

$$\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx = \frac{\pi}{2}(\pi - 2).$$

Example 1.33. *Evaluate* $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx.$

Solution. Let $$I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$$

Then $$I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x} dx \quad [\because \sin(\pi - x) = \sin x, \cos(\pi - x) = -\cos x]$$

[By Property (5)]

Adding the two values at I, we get

$$2I = \int_0^{\pi} \frac{(\pi - x + x)\sin x}{1+\cos^2 x}\,dx$$

$$= \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

$$= -\pi\left[\tan^{-1} \cos x\right]_0^{\pi}$$

$$= -\pi\left(-\frac{1}{4}\pi - \frac{1}{4}\pi\right)$$

$$2I = \frac{2}{4}\pi^2 \quad \Rightarrow \quad I = \frac{\pi^2}{4}$$

Therefore, $\displaystyle\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{4}$.

Example 1.34. *Show that* $\displaystyle\int_0^{\pi/2} \cos^3 2x \,.\, \sin^4 4x\, dx = 0.$

Solution. Let $\displaystyle I = \int_0^{\pi/2} \cos^3 2x \,.\, \sin^4 4x\, dx$

Putting $2x = t$, we get

$$I = \frac{1}{2}\int_0^{\pi} \cos^3 t \,.\, \sin^4 2t\, dt$$

$$= \frac{1}{2}\int_0^{\pi} 2^4 \,.\, \cos^3 t \,.\, \sin^4 t \,.\, \cos^4 t\, dt$$

$$= 8\int_0^{\pi} \sin^4 t \,.\, \cos^7 t\, dt = 0 \qquad \text{[By Property (7)]}$$

Example 1.35. *Evaluate* $\displaystyle\int_0^1 \cot^{-1}(1 - x + x^2)\, dx.$

Solution. The given integral can be written as

$$I = \int_0^1 \tan^{-1}\left(\frac{1}{1 - x + x^2}\right) dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{1}{1 + x(x-1)}\right) dx$$

$$= \int_0^1 \tan^{-1}\left\{\frac{x - (x-1)}{1 + x(x-1)}\right\} dx$$

$$= \int_0^1 [\tan^{-1} x - \tan^{-1}(x-1)]\, dx$$

$$= \int_0^1 \tan^{-1} x\, dx - \int_0^1 \tan^{-1}(x-1)\, dx$$

But $\displaystyle\int_0^1 \tan^{-1}(x-1)\, dx = \int_0^1 \tan^{-1}(1 - x - 1)\, dx$

$$= -\int_0^1 \tan^{-1} x\, dx \qquad \text{[By Property (5)]}$$

$$\therefore \qquad I = 2\int_0^1 \tan^{-1} x\, dx$$

Integrating it by parts, taking '1' as a second function, we have

$$I = 2\left[\tan^{-1} x \,.\, x\right]_0^1 - 2\int_0^1 \frac{x}{1+x^2}\, dx$$

$$= 2\,.\,1\,.\,\frac{\pi}{4} - \left[\log(1+x^2)\right]_0^1$$

$$= \frac{\pi}{2} - \log 2$$

Therefore,

$$\int_0^1 \cot^{-1}(1-x+x^2)\, dx = \frac{\pi}{2} - \log 2\,.$$

EXERCISE 1.3

1. Evaluate the given definite integrals as limit of sums:

i. $\int_a^b x\, dx$

ii. $\int_0^5 (x+1)\, dx$

iii. $\int_2^3 x^2\, dx$

iv. $\int_1^4 (x^2 - x)\, dx$

v. $\int_{-1}^1 e^x dx$

2. Evaluate the definite integrals:

i. $\int_{-1}^1 (x+1)\, dx$

ii. $\int_1^2 (4x^3 - 5x^2 + 6x + 9)\, dx$

iii. $\int_0^{\pi/2} \cos 2x\, dx$

iv. $\int_0^{\pi/4} \tan x\, dx$

v. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

vi. $\int_2^3 \frac{dx}{x^2-1}$

vii. $\int_2^3 \frac{x\, dx}{x^2+1}$

viii. $\int_0^1 x\, e^{x^2}\, dx$

ix. $\int_0^{\pi/4} (2\sec^2 x + x^3 + 2)\, dx$

x. $\int_0^1 \left(x\, e^x + \sin\frac{\pi x}{4}\right) dx$

3. Evaluate $\int_0^{\pi} \log(1+\cos x)\, dx.$

4. Evaluate $\int_0^1 \frac{\log(1+x)}{1+x^2}\, dx.$

5. Evaluate $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}\, dx.$

6. Show that $\int_0^{\pi} x\, f(\sin x)\, dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\, dx.$

7. Evaluate $\int_0^{\pi} \sin^6 x \cos^7 x\, dx.$

Answers

1. i. $\frac{a}{\sqrt{2}}$ ii. $\frac{35}{2}$ iii. $\frac{19}{3}$ iv. $\frac{27}{2}$
   v. $e-\frac{1}{e}$

2. i. 2 ii. $\frac{64}{3}$ iii. 0 iv. $\frac{1}{2}\log 2$
   v. $\frac{\pi}{2}$ vi. $\frac{1}{2}\log\frac{3}{2}$ vii. $\frac{1}{2}\log 2$ viii. $\frac{1}{2}(e-1)$
   ix. $\frac{\pi^4}{1024}+\frac{\pi}{2}+2$ x. $1+\frac{4}{\pi}-\frac{2\sqrt{2}}{\pi}$

3. $-\pi\log 2$ **4.** $\frac{\pi}{8}\log 2$ **5.** $\frac{1}{\sqrt{2}}\log(1+\sqrt{2})$ **7.** 0

1.2.5 Improper Integral

The integral $\int_a^b f(x)\,dx$ is an improper integral if

i. either the interval of integration $[a, b]$ is not finite *i.e.* either 'a' or 'b' or both 'a' and 'b' are infinite.
ii. or the integrand $f(x)$ is not bounded on $[a, b]$.
iii. neither the interval $[a, b]$is finite nor $f(x)$ is bounded over it.

1.2.6 Types of Improper Integral

Improper integral is of three types, which are explained as follows:

a. **Improper Integral of First Kind:**

The definite integral $\int_a^b f(x)\,dx$ is an improper integral of first kind if either 'a' or 'b' or both are infinite but $f(x)$ is bounded.

For example: $\int_1^\infty \frac{dx}{\sqrt{x}}, \int_{-\infty}^0 e^{2x}\,dx$ are improper integrals of first kind.

In this case, we define

i. $\int_a^\infty f(x)\,dx = \underset{t\to\infty}{\text{Lt}} \int_a^t f(x)\,dx, (t > a)$

The improper integral $\int_a^\infty f(x)\,dx$ will be convergent if the limit on the right hand side exists finitely and will be divergent if the limit is $+\infty$ or $-\infty$.

ii. $\int_{-\infty}^b f(x)\,dx = \underset{t\to-\infty}{\text{Lt}} \int_t^b f(x)\,dx \quad (t < b)$

The improper integral $\int_{-\infty}^{b} f(x)\,dx$ will be convergent if the limit on the right hand side exists finitely and will be divergent if the limit is + ∞ or – ∞.

iii. $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$ [For every c]

$$= \underset{t_1 \to -\infty}{\text{Lt}} \int_{t_1}^{c} f(x)\,dx + \underset{t_2 \to \infty}{\text{Lt}} \int_{c}^{t_2} f(x)\,dx \quad [t_1 < c < t_2]$$

The improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ is said to be convergent if the limit on the right hand side exists finitely and will be divergent if the limit is + ∞ or – ∞.

SOME SOLVED EXAMPLES

Example 1.36. *Examine the convergence of the improper integral* $\int_{1}^{\infty} \frac{dx}{\sqrt{x}}$.

Solution. Here
$$\int_{1}^{\infty} \frac{dx}{\sqrt{x}} = \lim_{t \to \infty} \int_{1}^{t} \frac{dx}{\sqrt{x}}$$
$$= \lim_{t \to \infty} \int_{1}^{t} x^{-1/2}\,dx$$
$$= \lim_{t \to \infty} \left[2x^{1/2}\right]_{1}^{t}$$
$$= \lim_{t \to \infty} [2\sqrt{t} - 2] = \infty$$

Hence, the given improper integral is divergent.

Example 1.37. *Solve the improper integral* $\int_{-\infty}^{0} e^{-x}\,dx$.

Solution. Here
$$\int_{-\infty}^{0} e^{-x}\,dx = \lim_{t \to -\infty} \int_{t}^{0} e^{-x}\,dx$$
$$= \lim_{t \to -\infty} \left[\frac{e^{-x}}{-1}\right]_{t}^{0}$$
$$= \lim_{t \to -\infty} [-(1 - e^{-t})]$$
$$= -1 + e^{\infty} = \infty$$

Hence, the given improper integral is divergent.

Example 1.38. *Check the convergence of the following improper integral* $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

Solution. Here
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$
$$= \lim_{t_1 \to -\infty} \int_{t_1}^{0} \frac{dx}{1+x^2} + \lim_{t_2 \to \infty} \int_{0}^{t_2} \frac{dx}{1+x^2}$$

$$= \lim_{t_1 \to -\infty} \left[\tan^{-1} x\right]_{t_1}^{0} + \lim_{t_2 \to \infty} \left[\tan^{-1} x\right]_{0}^{t_2}$$

$$= \lim_{t_1 \to -\infty} \left[\tan^{-1} 0 - \tan^{-1} t_1\right] + \lim_{t_2 \to \infty} \left[\tan^{-1} t_2 - \tan^{-1} 0\right]$$

$$= -[\tan^{-1}(-\infty)] + [\tan^{-1}(\infty)] = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Hence the given improper integral is convergent.

b. **Improper Integral of Second Kind:**

The definite integral $\int_a^b f(x)\,dx$ is said to be improper integral of second kind if both 'a' and 'b' are finite and $f(x)$ is not bounded (*i.e.* $f(x)$ has one or more points of infinite discontinuity).

For example: $\int_0^1 \frac{1}{x}\,dx, \int_1^4 \frac{dx}{(x-1)(x-4)}$ are improper integrals of second kind. In this case, we define

i. $\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f(x)\,dx$, if '$a$' is the only point of infinite discontinuity of $f(x)$.

If the limit on the R.H.S. exists finitely, then it is convergent otherwise it is divergent.

ii. $\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f(x)\,dx$, if '$b$' is the only point of infinite discontinuity of $f(x)$.

If the limit on the R.H.S. exists finitely, then it is convergent, otherwise, it is divergent.

iii. If $f(x)$ becomes infinite at some point 'c' only with $a < c < b$, then

$$\int_a^b f(x)\,dx = \lim_{\varepsilon_1 \to 0^+} \int_a^{c-\varepsilon_1} f(x)\,dx + \lim_{\varepsilon_2 \to 0^+} \int_{c+\varepsilon_2}^b f(x)\,dx$$

In general, if $c_1, c_2, c_3, \ldots\ldots, c_n$ are some finite number of points of infinite discontinuity of $f(x)$ on $[a, b]$, where $a < c_1 < c_2 < c_3, \ldots\ldots < c_{n-1} < c_n < b$, then

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \ldots\ldots + \int_{c_n}^b f(x)\,dx$$

If the limit on R.H.S. exists finitely, then the improper integral will be convergent otherwise it will be divergent.

SOME SOLVED EXAMPLES

Example 1.39. *Test the convergence of the integral* $\int_0^1 \frac{dx}{\sqrt{x}}$.

Solution. The given integral is of second kind and '0' is the point of infinite discontinuity on [0, 1].

Therefore,
$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0^+} \int_{0+\varepsilon}^1 x^{-1/2}\,dx$$

$$= \lim_{\varepsilon \to 0^+} \left[2\sqrt{x}\right]_{\varepsilon}^{1}$$

$$= \lim_{\varepsilon \to 0^+} 2(1-\sqrt{\varepsilon}) = 2 \text{ (finite)}$$

Hence given integral is convergent and converges to 2.

Example 1.40. *Test the convergence of the integral* $\int_0^1 \frac{dx}{x^2-3x+2}$

Solution. The given integral is of second kind and '1' is the only point of discontinuity of $f(x)$. Therefore, by definition,

$$
\begin{aligned}
\int_0^1 \frac{dx}{x^2-3x+2} &= \lim_{\varepsilon\to 0^+}\int_0^{1-\varepsilon}\frac{dx}{x^2-3x+2} \\
&= \lim_{\varepsilon\to 0^+}\int_0^{1-\varepsilon}\frac{dx}{(1-x)(2-x)} \\
&= \lim_{\varepsilon\to 0^+}\int_0^{1-\varepsilon}\left[\frac{1}{1-x}-\frac{1}{2-x}\right] \qquad \text{[By Partial fraction]} \\
&= \lim_{\varepsilon\to 0^+}\left[-\log(1-x)+\log(2-x)\right]_0^{1-\varepsilon} \\
&= \lim_{\varepsilon\to 0^+}\left[-\log\varepsilon+\log(1+\varepsilon)-\log 2\right] \\
&= \lim_{\varepsilon\to 0^+}\left[\log\frac{1+\varepsilon}{\varepsilon}-\log 2\right] \\
&= \lim_{\varepsilon\to 0^+}\left[\log\left(1+\frac{1}{\varepsilon}\right)-\log 2\right] \\
&= \log(\infty)-\log(2)=\infty
\end{aligned}
$$

Hence, the given integral is divergent.

Example 1.41. *Test the convergence of the integral* $\int_{-1}^1 \frac{dx}{x^2}$.

Solution. The given integral is of second kind and '0' is the only point of infinite discontinuity in [–1, 1].

Therefore, by definition,

$$
\begin{aligned}
\int_{-1}^1 \frac{dx}{x^2} &= \int_{-1}^0 \frac{dx}{x^2}+\int_0^1\frac{dx}{x^2} \\
&= \lim_{\varepsilon_1\to 0^+}\int_{-1}^{0-\varepsilon_1}\frac{dx}{x^2}+\lim_{\varepsilon_2\to 0^+}\int_{\varepsilon_2}^1\frac{dx}{x^2} \\
&= \lim_{\varepsilon_1\to 0^+}\left[\frac{x^{-2+1}}{-1}\right]_{-1}^{-\varepsilon_1}+\lim_{\varepsilon_2\to 0^+}\left[\frac{x^{-2+1}}{-1}\right]_{\varepsilon_2}^1 \\
&= \lim_{\varepsilon_1\to 0^+}\left[-x^{-1}\right]_{-1}^{-\varepsilon_1}+\lim_{\varepsilon_2\to 0^+}\left[-x^{-1}\right]_{\varepsilon_2}^1 \\
&= \lim_{\varepsilon_1\to 0^+}\left[\frac{1}{\varepsilon_1}-1\right]+\lim_{\varepsilon_2\to 0^+}\left[-1+\frac{1}{\varepsilon_2}\right] \\
&= \infty
\end{aligned}
$$

Hence, the given integral is divergent.

c. **Improper Integral of third kind (or Mixed kind):**

The definite integral $\int_a^b f(x)\,dx$ is said to be improper integral of third kind if either 'a' or 'b' or both are infinite and $f(x)$ is also unbounded.

For example: $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx, \int_{-\infty}^\infty \frac{1}{1-x}\,dx$ are improper integrals of third kind.

EXERCISE 1.4

1. Examine the convergence of the following improper integrals and if convergent, find their values:

i. $\int_2^\infty \frac{dx}{x\log x}$ ii. $\int_a^\infty \frac{x\,dx}{1+x^2}$ iii. $\int_{-\infty}^0 \frac{dx}{p^2+q^2x^2}$ iv. $\int_{\sqrt{2}}^\infty \frac{dx}{x\sqrt{x^2-1}}$

v. $\int_1^\infty \frac{\tan^{-1}x}{x^2}\,dx$ vi. $\int_0^\infty \cos x\,dx$

2. Examine the convergence of the following improper integrals and if convergent, find their values also.

i. $\int_1^2 \frac{x}{\sqrt{x-1}}\,dx$ ii. $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$ iii. $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}}\,dx$ iv. $\int_0^4 \frac{dx}{x(4-x)}$

v. $\int_0^\pi \frac{dx}{\sin x}$ vi. $\int_0^1 \log x\,dx$

Answers

1. i. divergent ii. divergent iii. convergent; $\frac{\pi}{2pq}$ iv. convergent; $\frac{\pi}{4}$

v. convergent; $\frac{\pi}{4}+\frac{1}{2}\log 2$ vi. divergent

2. i. convergent; $\frac{8}{3}$ ii. convergent; $\frac{\pi}{3}$ iii. convergent; 2 iv. divergent

v. divergent vi. convergent; –1

1.2.7 Comparison tests for convergence of $\int_a^b f(x)\,dx$ at 'a'

1.2.7.1 Comparison Test-I

Statement: If f and g are two positive functions such that $f(x) \le g(x)$ for all x in $(a, b]$ and 'a' is the only infinite discontinuity on $[a, b]$, then

i. $\int_a^b g\,dx$ is convergent $\Rightarrow \int_a^b f\,dx$ is convergent

ii. $\int_a^b fdx$ is divergent $\Rightarrow \int_a^b gdx$ is divergent

Proof. Since $0 < f(x) \le g(x) \ \forall \ x \in (a, b]$

$$\therefore \qquad \int_{a+\in}^b fdx \le \int_{a+\in}^b gdx \quad \text{for } 0 < \varepsilon < b - a \qquad ...(1)$$

i. Let $\int_a^b gdx$ be convergent at 'a', then there exists a positive number M such that

$$\int_{a+\in}^b gdx < M \quad \text{for } 0 < \varepsilon < b - a \qquad ...(2)$$

From (1) and (2), we get

$$\int_{a+\in}^b fdx < M \quad \text{for } 0 < \varepsilon < b - a$$

$\Rightarrow \quad \int_a^b fdx$ is convergent at 'a'.

ii. Let $\int_a^b fdx$ be divergent at 'a'. Then $\int_{a+\in}^b fdx$ is unbounded above and hence from (1), $\int_{a+\in}^b gdx$ is unbounded above.

Hence, $\int_a^b gdx$ is divergent at a.

1.2.7.2 Comparison Test-II

Statement: If 'f' and 'g' are two positive functions on $(a, b]$, 'a' being the only point of infinite discontinuity such that

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = l \neq 0$$

then the two integrals $\int_a^b fdx$ and $\int_a^b gdx$ converges or diverges together at 'a'.

Proof: As f, g are positive in $(a, b]$, therefore $\dfrac{f(x)}{g(x)} > 0 \quad \forall \ x \in (a, b]$

Hence $\qquad \lim_{x \to a^+} \dfrac{f(x)}{g(x)} = l \ge 0$

But $l \neq 0$ (given). Therefore $l > 0$.

Now choose a positive number ε such that $l - \varepsilon > 0$,

since $\qquad \lim_{x \to a^+} \dfrac{f(x)}{g(x)} = l$

$\Rightarrow$ There exists a neighbourhood (a, c) of a $(a < c < d)$ such that

$$\left| \frac{f(x)}{g(x)} - l \right| < \in, \forall \ x \in (a, c]$$

$$\Rightarrow \qquad -\varepsilon < \frac{f(x)}{g(x)} - l < \varepsilon, \quad \forall \ x \in (a, c]$$

$$\Rightarrow \qquad (l - \varepsilon)\, g(x) < f(x) < (l + \varepsilon)\, g(x) \quad \forall \ x \in (a, c]$$

Case I: Let $\int_a^b fdx$ converges to a:

$\Rightarrow \quad \int_a^c fdx$ converges at a. [$\because \quad a < c < b$, and $\int_c^b fdx$ is a proper integral]

$\Rightarrow \quad (l - \varepsilon) \int_a^c gdx$ converges at a. [By comparison test I]

$\Rightarrow \quad \int_a^b gdx$ converges at a.

Case II: Let $\int_a^b f(x)$ diverges at a:

$\Rightarrow \quad \int_a^c f(x)(dx)$ diverges at a. [$\because \quad a < c < b$ and $\int_c^b fdx$ is a proper integral]

$\Rightarrow \quad (l + \varepsilon) \int_a^c gdx$ diverges at a [By comparison test I]

$\Rightarrow \quad \int_a^b gdx$ diverges at a.

Similarly, it can be shown that if $\int_a^b gdx$ converges at 'a', then $\int_a^b fdx$ converges at 'a' and if $\int_a^b gdx$ diverges at 'a', then $\int_a^b fdx$ diverges at 'a'

Hence the theorem.

1.2.8 Important Theorems

i. The improper integral $\int_a^b \frac{dx}{(x-a)^n}$ is convergent if and only if $n < 1$.

ii. The improper integral $\int_a^b \frac{dx}{(b-x)^n}$ is convergent if and only if $n < 1$.

Proof: i. The given integral $\int_a^b \frac{dx}{(x-a)^n}$ is a proper integral for $n \le 0$ and therefore convergent. If $n > 0$, then it is an improper integral and 'a' is only point of infinite discontinuity.

Case I: If $n \neq 1$

$$\begin{aligned}
\int_a^b \frac{dx}{(x-a)^n} &= \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b \frac{dx}{(x-a)^n}, \; 0 < \varepsilon < b - a \\
&= \lim_{\varepsilon \to 0^+} \left[\frac{(x-a)^{-n+1}}{-n+1} \right]_{a+\varepsilon}^b \\
&= \lim_{\varepsilon \to 0^+} \left[\frac{1}{(b-a)^{n-1}} - \frac{1}{\varepsilon^{n-1}} \right] \frac{1}{(1-n)} \\
&= \begin{cases} \dfrac{1}{(1-n)(b-a)^{n-1}}, & \text{finite if } n < 1 \\ \infty & \text{if } n > 1 \end{cases}
\end{aligned}$$

$\therefore \quad \int_a^b \frac{dx}{(x-a)^n}$ converges for $0 < n < 1$ and diverges for $n > 1$

Case II. If $n = 1$

$$\int_a^b \frac{dx}{(x-a)^n} = \int_a^b \frac{dx}{x-a} = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b \frac{dx}{x-a}$$

$$= \lim_{\varepsilon \to 0^+} [\log|x-a|]_{a+\varepsilon}^b$$

$$= \lim_{\varepsilon \to 0^+} [\log(b-a) - \log \varepsilon]$$

$$= \lim_{\varepsilon \to 0^+} \left(\log \frac{b-a}{\varepsilon}\right) = \infty$$

$\therefore \quad \int_a^b \frac{dx}{(x-a)^n}$ diverges if $n = 1$

Hence, $\int_a^b \frac{dx}{(x-a)^n}$ is convergent for $n < 1$ and divergent for $n \geq 1$.

ii. The given integral $\int_a^b \frac{dx}{(b-x)^n}$ is a proper integral for $n \leq 0$ and therefore converges. If $n > 0$, then it is an improper integral and 'b' is the only point of infinite discontinuity.

Case I: If $n \neq 1$

$$\int_a^b \frac{dx}{(b-x)^n} = \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} \frac{dx}{(b-x)^n}$$

$$= \lim_{\varepsilon \to 0^+} \left[\frac{(b-x)^{-n+1}}{-(-n+1)}\right]_a^{b-\varepsilon}$$

$$= \lim_{\varepsilon \to 0^+} \frac{1}{n-1} [(b-x)^{-n+1}]_a^{b-\varepsilon}$$

$$= \lim_{\varepsilon \to 0^+} \frac{1}{n-1} [\varepsilon^{1-n} - (b-a)^{1-n}]$$

$$= \begin{cases} \infty & \text{if } n > 1 \\ \dfrac{1}{(1-n)(b-a)^{n-1}} & \text{if } n < 1 \end{cases}$$

$\therefore \quad \int_a^b \frac{dx}{(b-x)^n}$ converges for $0 < n < 1$ and diverges for $n > 1$.

Case II: If $n = 1$

$$\int_a^b \frac{dx}{(b-x)^n} = \int_a^b \frac{dx}{b-x}$$

$$= \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} \frac{dx}{b-x}$$

$$= \lim_{\varepsilon \to 0^+} [-\log|b-x|]_a^{b-\varepsilon}$$

$$= \lim_{\varepsilon \to 0^+} [-\log \varepsilon + \log(b-a)] = \infty \qquad [\because \quad \log 0 = -\infty]$$

$\therefore \quad \int_a^b \frac{dx}{(b-x)^n}$ diverges if $n = 1$

Hence, $\int_a^b \frac{dx}{(b-x)^n}$ is convergent for $n < 1$ and divergent for $n \geq 1$

Remark: $\int_0^1 \frac{1}{x^n} dx$ is convergent if $n < 1$ and divergent if $n \geq 1$.

SOME SOLVED EXAMPLES

Example 1.42. *Examine the convergence of the integral* $\int_0^1 \frac{dx}{x^{1/2}(1+x^2)}$

Solution. Let $$I = \int_0^1 \frac{dx}{x^{1/2}(1+x^2)}$$

Here, $f(x) = \frac{1}{x^{1/2}(1+x^2)}$ and 0 is only point of infinite discontinuity of $f(x)$ and $f(x) > 0$ in $(0, 1]$

Take $$g(x) = \frac{1}{x^{1/2}}$$

Now, $$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} \frac{1}{1+x^2} = 1$$ [which is finite and non-zero]

$\therefore$ By comparison test,
the integrals

$\int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ converges or diverges together.

But the integral $\int_0^1 g(x)\,dx = \int_0^1 \frac{1}{x^{1/2}}$ is convergent at $x = 0$ $\left[\because \; n = \frac{1}{2} < 1\right]$

$\therefore$ $\int_0^1 f(x)\,dx = \int_0^1 \frac{dx}{x^{1/2}(1+x^2)}$ is also convergent.

Example 1.43. *Discuss the convergence of the integral* $\int_0^{\pi/2} \frac{\sin x}{x^{3/2}} dx$.

Solution. Let $$I = \int_0^{\pi/2} \frac{\sin x}{x^{3/2}} dx$$

Here, $f(x) = \frac{\sin x}{x^{3/2}}$ and '0' is the only point of infinite discontinuity of $f(x)$ and $f(x) > 0$ in $\left[0, \frac{\pi}{2}\right]$

Take $$g(x) = \frac{1}{x^{1/2}}$$

Now, $$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} \frac{\sin x}{x} = 1 \neq 0, \infty.$$

$\therefore$ By comparison test, the integrals $\int_0^{\pi/2} f(x)\,dx$ and $\int_0^{\pi/2} g(x)\,dx$ converge or diverge together.

But $\int_0^{\pi/2} g(x)\,dx = \int_0^{\pi/2} \frac{dx}{\sqrt{x}}$ $\left[\because \ n = \frac{1}{2} < 1\right]$ is convergent at $x = 0$

i.e., $\int_0^{\pi/2} f(x)\,dx = \int_0^{\pi/2} \frac{\sin x}{x^{3/2}}\,dx$ is also convergent at $x = 0$

1.2.9 Comparison Test for Convergence at ∞

1.2.9.1 Comparison Test I

If f and g are two positive functions such that $f(x) \le g(x)$ for all $x \ge a$, then

i. $\int_a^\infty f dx$ converges if $\int_a^\infty g dx$ converges

ii. $\int_a^\infty g dx$ diverges if $\int_a^\infty f dx$ diverges

Proof: Here f and g are two positive functions such that $f(x) \le g(x)$ for all $x \in [a, t]$

$$\therefore \quad \int_a^t f(x)\,dx \le \int_a^t g\,dx \qquad ...(1)$$

i. Let $\int_a^\infty g dx$ be convergent, so that there exists a positive number M.

such that
$$\int_a^t g dx < M \ \forall\ t \ge a \qquad ...(2)$$

From (1) and (2), we have

$$\int_a^t f dx < M \ \forall\ t \ge a$$

Hence $\int_a^\infty f dx$ is convergent.

ii. Let $\int_a^\infty f dx$ be divergent

$\Rightarrow$ $\int_a^t f dx$ is not bounded above and hence from (1), $\int_a^t g dx$ is also not bounded above, consequently, $\int_a^\infty g dx$ is divergent.

1.2.9.2 Comparison Test II

If f and g are two positive functions on $[a, \infty)$ such that

i. $$\lim_{x\to\infty} \frac{f(x)}{g(x)} = l$$ (where l is non-zero and finite),

then the two integrals $\int_a^\infty f dx$ and $\int_a^\infty g dx$ converge or diverge together

Proof. i. As $\frac{f(x)}{g(x)} > 0$ for all $x \ge a$ and $l \ne 0$ $\quad$ [$\because$ $f(x)$ and $g(x)$ are positive functions]

$\therefore \quad l > 0$

Choose $\varepsilon > 0$ such that $l - \varepsilon > 0$

Since $\lim_{x\to\infty} \frac{f(x)}{g(x)} = l$, therefore there exists a number k such that

$$\left|\frac{f(x)}{g(x)} - l\right| < \varepsilon \ \forall\ x \geq k$$

$$\Rightarrow \qquad l - \varepsilon < \frac{f(x)}{g(x)} < l + \varepsilon \ \forall\ x \geq k > a$$

or $$(l - \varepsilon)\, g(x) < f(x) < (l + \varepsilon)\, g(x) \ \forall\ x \geq k > a$$

By comparison test I, if $\int_a^\infty f dx$ converges, then $\int_a^\infty g(x)\,dx$ converges and if $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.

Similarly, divergence of one implies the divergence of other.

Hence, the two integrals $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ converge or diverge together.

1.2.10 Important Theorem

Statement: The improper integral $\int_a^\infty \frac{dx}{x^n}$, $(a > 0)$ converges if and only if $n > 1$ and diverges for $n \leq 1$.

Proof:

$$\int_a^\infty \frac{dx}{x^n} = \lim_{t\to\infty} \int_a^t \frac{dx}{x^n}$$

$$= \lim_{t\to\infty} \left[\frac{x^{-n+1}}{-n+1}\right]_a^t, \text{ if } n \neq 1$$

$$= \lim_{t\to\infty} \left[\frac{t^{1-n}}{1-n} - \frac{a^{1-n}}{1-n}\right], \text{ if } n \neq 1$$

$$= \begin{cases} \dfrac{-a^{1-n}}{1-n}, & \text{(which is finite) if } n > 1 \\ \infty & \text{if } n < 1 \end{cases}$$

Also when $n = 1$, we have

$$\int_a^\infty \frac{1}{x^n}\,dx = \int_a^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_a^t \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} [\log x]_a^t$$

$$= \lim_{t\to\infty} [\log t - \log a]$$

$$= \log \infty - \log a = \infty$$

Hence, $\int_a^\infty \frac{dx}{x^n}$ converges if and only if $n > 1$ and diverges for $n \leq 1$.

Remark: $\int_a^\infty \frac{dx}{x^n} dx$ is convergent if $(a > 0)$ is convergent if $n > 1$ and divergent if $n \leq 1$.

SOME SOLVED EXAMPLES

Example 1.44. *Test the convergence of the integral* $\int_1^{\infty} \frac{x^3}{(1+x)^5}\, dx.$

Solution. Let $$I = \int_1^{\infty} \frac{x^3}{(1+x)^5}\, dx$$

Here, $$f(x) = \frac{x^3}{(1+x)^5} = \frac{x^3}{x^5\left(1+\frac{1}{x}\right)^5}$$

$\Rightarrow$ $$f(x) = \frac{1}{x^2\left(1+\frac{1}{x}\right)^5}$$

Take $$g(x) = \frac{1}{x^2}$$

Now, $$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{1}{\left(1+\frac{1}{x}\right)^5} = 1 \neq 0, \infty$$

$\therefore$ By comparison test, the integrals $\int_1^{\infty} f(x)\, dx$ and $\int_1^{\infty} g(x)\, dx$ converge or diverge together.

But the integral $\int_1^{\infty} g(x)\, dx = \int_1^{\infty} \frac{1}{x^2}\, dx$ [$\because$ $n = 2 > 1$

is convergent.

$\therefore$ The integral $\int_1^{\infty} \frac{x^3}{(1+x)^5}\, dx$ is convergent.

Example 1.45. *Discuss the convergence of the improper integral* $\int_1^{\infty} x^n \,.\, e^{-x}\, dx.$

Solution. Let $$I = \int_1^{\infty} x^n \,.\, e^{-x}\, dx$$

Here, $$f(x) = x^n \,.\, e^{-x}$$

Take $$g(x) = \frac{1}{x^2}$$

Now, $$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{x^{n+2}}{e^x} = 0 \ \forall\ n$$

Now, $\int_1^{\infty} g(x)\, dx = \int_1^{\infty} \frac{dx}{x^2}$ is convergent. [$\because$ $n = 2 > 1$]

$\therefore$ By comparison test, $\int_1^{\infty} x^n \,.\, e^{-x} dx$ is also convergent.

1.2.11 Absolute Convergence

The improper integral $\int_a^b f dx$ is said to be absolutely convergent if $\int_a^b |f|\,dx$ is convergent.

Example 1.46. *Test the convergence at* $\int_0^1 \frac{\sin 1/x}{\sqrt{x}}\,dx$.

Solution. Let $$I = \int_0^1 \frac{\sin 1/x}{\sqrt{x}}\,dx$$

Here, $$f(x) = \frac{\sin 1/x}{\sqrt{x}},$$

does not keep the same sign in the neighbourhood of '0' and '0' is the point of infinite discontinuity of 'f' in [0, 1].

Now, $$|f(x)| = \left|\frac{\sin 1/x}{\sqrt{x}}\right| = \frac{|\sin 1/x|}{\sqrt{x}} \le \frac{1}{\sqrt{x}} = g(x) \text{ (say)} \qquad \left[\because \left|\sin\frac{1}{x}\right| \le 1\right]$$

But $$\int_0^1 g(x)\,dx = \int_0^1 \frac{1}{\sqrt{x}}\,dx \text{ is convergent at } x = 0. \qquad \left[\because n = \frac{1}{2} < 1\right.$$

$\therefore$ $\int_0^1 |f|$ is convergent at 0.

Hence the given integral $\int_0^1 \frac{\sin 1/x}{\sqrt{x}}\,dx$ converges absolutely at '0'.

EXERCISE 1.5

1. Discuss the convergence of the following integrals:

i. $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ ii. $\int_0^1 \frac{x^n}{1+x}\,dx$ iii. $\int_0^1 \frac{dx}{x^3(1+x^2)^5}$

2. Test the convergence of $\int_0^1 \frac{\log x}{\sqrt{1-x^2}}\,dx$

3. Examine the convergence of the following integrals:

i. $\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}$ ii. $\int_0^1 \frac{\log x}{\sqrt{x}}\,dx$ iii. $\int_1^2 \frac{\sqrt{x}}{\log x}\,dx$ iv. $\int_0^{\pi/4} \frac{dx}{\sqrt{\sin x}}$

4. Test the convergence of the following integrals:

i. $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$ ii. $\int_1^\infty \frac{\log x}{x^2}\,dx$ iii. $\int_1^\infty \frac{dx}{\sqrt{x}\,(1+x^n)}$ iv. $\int_1^\infty \frac{dx}{\sqrt{x}\,(2+x)}$

v. $\int_0^\infty e^{-x^2}\,dx$

[**Hint:** $e^{x^2} > x^2 \ \forall\ x \in \mathbb{R}$]

Answers

1.	i. convergent	ii. convergent if $n > -1$		iii. divergent
2.	convergent			
3.	i. convergent	ii. convergent	iii. convergent	iv. convergent
4.	i. convergent	ii. convergent	iii. convergent, if $n > 1/2$	
	iv. convergent	v. convergent		

INTERESTING FACTS

- This concept is used in pharmacological research to find out the plasma drug concentration, that is what is the maximum drug concentration and when it occurs.
- The '**R**'-value of any drug, which calculates the ratio of two different quantities is measured using this concept.
- Engineers use integral to find the ***Centre of Mass*** of any object.
- An interesting relationship in calculus is that the derivative and the integral are inverse processes. They are reverse of each other and they are linked using "***The Fundamental Theorem of Calculus***".

VIDEO REFERENCES

Beta & Gamma Function

Improper Integrals (Cont.) 1

Improper Integrals (Cont.) 2

APPLICATIONS TO REAL LIFE

- Application in statistics and probability.
- Significance in quantum physics and economics, which is created on the basis of probability distribution
- It is even used to find average changes, volumes, error estimations and surface areas.
- The same concept is used in finding Kinetic energy as well.

HISTORY

Special functions occur quite frequently in mathematical analysis. Among the special functions, gamma function seemed to be widely used. The gamma function $\Gamma(x)$ is applied in exact sciences almost as the well known factorial symbol $x!$. It was introduced by the famous mathematician L. Euler (1729). Beta function was first studied by Euler and Legendre and was given its name by Jacques Binet.

1.3 BETA, GAMMA FUNCTIONS AND THEIR PROPERTIES

1.3.1 Gamma Function

For $n > 0$, the improper integral $\int_0^\infty e^{-x} x^{n-1}\, dx$ is defined as Gamma function and denoted by $\Gamma(n)$ (read as gamma n). It is also known as Eulerian Integral of second kind. Thus,

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\, dx, n > 0 \qquad ...(1)$$

Note that Gamma function plays an important role in evaluation of definite integrals.

1.3.1.1 Properties of Gamma Function

a. To show that $\Gamma(n+1) = n\,\Gamma(n)$

We have, $\Gamma(n+1) = \int_0^{\infty} e^{-x} x^{(n+1)-1} dx$ [by (1)]

$$= [-e^{-x}x^n]_0^{\infty} + n\int_0^{\infty} e^{-x} x^{n-1} dx$$

$$= 0 + n\,\Gamma(n)$$

Thus, $\Gamma(n+1) = n\,\Gamma(n)$ ...(2)

b. To show that $\Gamma(n) = (n-1)!$, where n is positive integer.

We have $\Gamma(n) = (n-1)\,\Gamma(n-1)$ [by (2)]

$= (n-1)\,(n-2)\,\Gamma(n-2)$ [by (2)]

$= (n-1)\,(n-2)\,(n-3)\,...3.2\,\Gamma(2)$ [by repeated application of (2)]

$= (n-1)\,(n-2)\,(n-3)...3.2.1\,\Gamma(1)$

$= (n-1)!\,\Gamma(1) = (n-1)!\int_0^{\infty} e^{-x}\, dx$ [by (1)]

$= (n-1)!\,[-e^{-x}]_0^{\infty} = (n-1)!$

Thus $\Gamma(n) = (n-1)!$, n is positive integer

Remark 1: The formula $\Gamma(n+1) = n\,\Gamma(n)$ is called as recurrence formula for Gamma Function.
Remark 2: Note that $\Gamma(1) = 1$ can find out in (*b*).

c. To show that, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Put $n = \frac{1}{2}$ in (1), we obtain

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} e^{-x}\, x^{-1+\frac{1}{2}}\, dx$$

Putting $x = v^2 \Rightarrow dx = 2v\,dv$, we get

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^{\infty} e^{-v^2}\, dv \qquad ...(4)$$

Writing u in place of v in (4), we obtain

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^{\infty} e^{-u^2}\, du \qquad ...(5)$$

Multiplying (4) and (5), we get

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^{\infty}\int_0^{\infty} e^{-(u^2+v^2)}\, du\, dv$$

Now let $u = r\cos\theta$, $v = r\sin\theta$ then $u^2 + v^2 = r^2$ and $\theta = \tan^{-1}\left(\frac{v}{u}\right)$. Also $du\,dv = rdr\,d\theta$ and r varies from 0 to ∞; θ varies from 0 to $\frac{\pi}{2}$.

$$\therefore \quad \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} d\theta = 2\int_0^{\pi/2} d\theta = \pi$$

Hence, $$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \qquad ...(6)$$

d. To show that $\Gamma(0) = \infty$

From (2), we have $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$

As $n \to 0$, $\Gamma(0) = \lim\limits_{n\to 0}\dfrac{\Gamma(1)}{n} = \lim\limits_{n\to 0}\dfrac{1}{n} = \infty$

Thus, $$\Gamma(0) = \infty \qquad ...(7)$$

Further note that $\Gamma(-1), \Gamma(-2), \Gamma(-3)$ etc. are also undefined. Hence Gamma function is continuous for any $n > 0$ and is discontinuous at $n = 0, -1, -2, \ldots$.

Thus, $\Gamma(n)$ is defined for all n, except for zero and negative integers.

e. To show that $\Gamma(n+1) = (m+1)^{n+1}(-1)^n \int_0^1 x^m (\log x)^n\, dx$

where n is a positive integer and $m > -1$.

We have $\int_0^1 x^m (\log x)^n\, dx$

Put $x = e^{-y}$, then $dx = -e^{-y}\,dy = -x dy$

$$\therefore \quad \int_0^1 x^m (\log x)^n\, dx = \int_0^{\infty} e^{-my}(-y)^n e^{-y}\,dy = (-1)^n \int_0^{\infty} y^n e^{-(m+1)y}\,dy$$

Put $(m+1)y = u$, so, $dy = \dfrac{du}{m+1}$

$$\therefore \quad \int_0^1 x^m (\log x)^n\, dx = (-1)^n \int_0^{\infty}\frac{u^n}{(m+1)^n}\,.\,e^{-u}\,.\,\frac{du}{(m+1)}$$

$$= \frac{(-1)^n}{(m+1)^{n+1}}\int_0^{\infty} e^{-u}\,.\,u^{(n+1)-1}\,du$$

$$= \frac{(-1)^n}{(m+1)^{n+1}}\,\Gamma(n+1) \qquad \text{[by (1)]}$$

Thus, $$\Gamma(n+1) = (m+1)^{n+1}(-1)^n \int_0^1 x^m (\log x)^n\, dx \qquad ...(8)$$

where n is a positive integer and $m > -1$.

Imp. Formula: $\int_0^{\pi/2} \sin^m \theta \cos^n \theta \, d\theta = \dfrac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}, m, n > -1.$

SOME SOLVED EXAMPLES

Example 1.47. *Evaluate* $\int_0^{\infty} x^{1/4} e^{-\sqrt{x}} \, dx$.

Solution. We have $\quad I = \int_0^{\infty} e^{-\sqrt{x}} x^{1/4} \, dx$

Now put $\quad \sqrt{x} = u \quad \Rightarrow \quad x = u^2 \quad \Rightarrow \quad dx = 2\, u du$

$\therefore \quad I = \int_0^{\infty} e^{-u} (u^2)^{1/4} \, 2u \, du = 2\int_0^{x} e^{-u} u^{3/2} \, du$

$$= 2\int_0^{\infty} e^{-u} u^{\frac{5}{2}-1} \, du = 2\Gamma\left(\frac{5}{2}\right)$$

$$= 2 \,.\, \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = 2 \,.\, \frac{3}{2} \,.\, \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \qquad [\because \quad \Gamma(n+1) = n\Gamma(n)]$$

$$= \frac{3}{2}\sqrt{\pi} \quad \textbf{Answer} \qquad \left[\because \ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

Example 1.48. *Evaluate* $\int_0^{\infty} x^{n-1} e^{-h^2x^2} \, dx$.

Solution. We have $\quad l = \int_0^{\infty} x^{n-1} e^{-h^2x^2} dx$

Putting $h^2x^2 = y$, so that

$$h^2 \, 2x dx = dy$$

$\therefore \quad dx = \dfrac{1}{2}\dfrac{dy}{h^2 x} = \dfrac{1}{2}\dfrac{dy}{h\sqrt{y}}$

$\therefore \quad I = \int_0^{\infty} e^{-y} \,.\left(\dfrac{\sqrt{y}}{h}\right)^{n-1} \dfrac{1}{2}\dfrac{dy}{h\sqrt{y}}$

$$= \frac{1}{2h^n}\int_0^{\infty} e^{-y} y^{\frac{n-2}{2}} \, dy$$

$$= \frac{1}{2h^n}\int_0^{\infty} e^{-y} y^{\frac{n}{2}-1} \, dy$$

$$= \frac{1}{2h^n}\Gamma\left(\frac{n}{2}\right) \quad \textbf{Answer} \qquad \text{[By definition of Gamma function]}$$

Example 1.49. *Evaluate* $\int_0^\infty x^6 e^{-2x} dx$.

Solution. We have $I = \int_0^\infty e^{-2x}\, x^6 dx$

Putting $2x = y \Rightarrow dx = \frac{dy}{2}$, then

$$I = \int_0^\infty e^{-y}\left(\frac{y}{2}\right)^6 \frac{dy}{2} = \frac{1}{2^7}\int_0^\infty e^{-y} y^{7-1} dy = \frac{1}{2^7}\Gamma(7)$$

$$I = \frac{1}{2^7}\Gamma(7) \qquad \text{[By definition]}$$

$$\Rightarrow \quad = \frac{1}{2^7} 6! = \frac{45}{8} \quad \textbf{Answer} \qquad [\because \quad \Gamma(n+1) = n!, n > 0]$$

Example 1.50. *Prove that* $\int_0^\infty e^{(2ax - x^2)}\, dx = \frac{1}{2}\sqrt{\pi}\,.\, e^{a^2}$.

Solution. We have $I = \int_0^\infty e^{(2ax - x^2)} dx = \int_0^\infty e^{a^2 - (x^2 - 2ax + a^2)}\, dx$

$$= \int_0^\infty e^{a^2 - (x-a)^2}\, dx = e^{a^2}\int_0^\infty e^{-(x-a)^2} dx$$

Putting $(x - a) = y \Rightarrow dx = dy$

$$\therefore \quad I = e^{a^2}\int_{-a}^\infty e^{-y^2}\, dy \qquad ...(1)$$

Now by the definition of gamma function

$$\Gamma(n) = \int_0^\infty e^{-u} u^{n-1} du, n > 0$$

Put $n = \frac{1}{2}$, we obtain

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-u} u^{-1/2}\, du$$

Now put $u = y^2 \Rightarrow du = 2y\, dy$, we get

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-y^2} (y^2)^{-1/2} \,.\, 2y dy \qquad \left[\because \; \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$\Rightarrow \quad \sqrt{\pi} = 2\int_0^\infty e^{-y^2} dy \qquad ...(2)$$

$$\Rightarrow \quad \int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2}$$

Using (2), (1) becomes $I = e^{a^2} \,.\, \frac{\sqrt{\pi}}{2} = \frac{1}{2}\sqrt{\pi}\,.\, e^{a^2}$ **Proved.**

Example 1.51. *Evaluate* $\int_0^{\infty} x^{-3/2}(1-e^{-x})\,dx$.

Solution. Let $$I = \int_0^{\infty} x^{-3/2}(1-e^{-x})\,dx$$

$$\Rightarrow \quad I = \left[(1-e^{-x})\frac{x^{-1/2}}{\left(\frac{-1}{2}\right)}\right]_0^{\infty} - \int_0^{\infty} e^{-x}\cdot\frac{x^{-1/2}}{\left(\frac{-1}{2}\right)}\,dx$$

$$= 0 + 2\int_0^{\infty} e^{-x}x^{-1/2}\,dx$$

$$= 2\int_0^{\infty} e^{-x}x^{\frac{1}{2}-1}\,dx = 2\Gamma\left(\frac{1}{2}\right) \quad \textbf{Answer} \qquad \text{[By definition]}$$

$$= 2\sqrt{\pi}.$$

1.3.2 Beta Function

The Beta function is denoted and defined as

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad \text{...(1)}$$

Where m, n are positive numbers, integer or fractional. This is also known as Eulerian integral of first kind.

1.3.2.1 Simple Properties of Beta Function

i. To show that: $B(m,n) = B(n,m)$ [Symmetry]

We have $$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \int_0^1 (1-x)^{m-1}[1-(1-x)]^{n-1}\,dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 (1-x)^{m-1}x^{n-1}\,dx = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

$$= B(n,m)$$

Thus, $B(m,n) = B(n,m)$

ii. To show that: $\int_0^a x^{m-1}(a-x)^{n-1}\,dx = a^{m+n-1}B(m,n)$.

From (1), we have $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = B(m,n)$

Putting $x = \dfrac{y}{a}$ $\Rightarrow$ $dx = \dfrac{dy}{a}$ and y varies from 0 to a.

$$\int_0^a \left(\frac{y}{a}\right)^{m-1}\left(1-\frac{y}{a}\right)^{n-1}\frac{dy}{a} = B(m,n)$$

$$\Rightarrow \quad \frac{1}{a^{m+n-1}}\int_0^a y^{m-1}(a-y)^{n-1}dy = B(m,n)$$

$$\Rightarrow \quad \int_0^a x^{m-1}(a-x)^{n-1}\,dx = a^{m+n-1}B(m,n)$$

iii. To show that $B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$

We know that $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

Putting $x = \frac{1}{y+1}$ so that $dx = -\frac{dy}{(1+y)^2}$ and y varies from ∞ to 0.

$$\therefore \quad B(m,n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left[1-\frac{1}{1+y}\right]^{n-1}\left[-\frac{dy}{(1+y)^2}\right]$$

$$\Rightarrow \quad B(m,n) = \int_0^\infty \frac{1}{(1+y)^{m-1}}\cdot\frac{y^{n-1}}{(1+y)^{n-1}}\cdot\frac{dy}{(1+y)^2}$$

$$B(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy \qquad \text{...(1)}$$

(In this integral m and n may be changed, by the virtue of symmetry of the function:)
Again, (1) may be written as

$$= \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}dy + \int_1^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy \qquad \text{...(2)}$$

In order to solve second integral on R.H.S., put $y = \frac{1}{x}$, so that $dy = -\frac{1}{x^2}dx$ and x varies from 1 to 0.

$$\therefore \quad \int_1^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy = \int_1^0 \left(\frac{1}{x}\right)^{n-1}\frac{x^{m+n}}{(1+x)^{m+n}}\left(-\frac{1}{x^2}\right)dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx \qquad \text{...(3)}$$

Using (3), (2) becomes

$$B(m,n) = \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}dy + \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx + \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(y)dy\right]$$

$$= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$$

Hence, $B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$

iv. To show that $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$

We know $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

Putting $x = \sin^2\theta \quad \Rightarrow \quad dx = 2\sin\theta\cos\theta\, d\theta$ and θ varies from 0 to $\frac{\pi}{2}$

$$\therefore \qquad B(m, n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

Thus
$$B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \,.\, \cos^{2n-1}\theta \,.\, d\theta$$

Particular case:

When $m = \frac{1}{2}$, $n = \frac{1}{2}$, above expression gives,

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = 2\int_0^{\pi/2} d\theta = 2 \,.\, \frac{\pi}{2} = \pi$$

v. Show that $(a - b)^{m+n-1} B(m, n) = \int_b^a (x-b)^{m-1}(a-x)^{n-1} dx$

We know that
$$B(m, n) = \int_0^1 y^{m-1}(1-y)^{n-1} dy$$

Let $y = \frac{x-b}{a-b} \quad \Rightarrow \quad dy = \frac{dx}{a-b}$ and y varies from b to a.

$$\therefore \qquad B(m, n) = \int_b^a \left(\frac{x-b}{a-b}\right)^{m-1}\left(1-\frac{x-b}{a-b}\right)^{n-1} \frac{dx}{a-b}$$

$$= \frac{1}{(a-b)^{m+n-1}} \int_b^a (x-b)^{m-1}(a-x)^{n-1} dx$$

$$\therefore \qquad \int_b^a (x-b)^{m-1}(a-x)^{n-1}\, dx = (a-b)^{m+n-1} B(m, n)$$

Thus,
$$\int_b^a (x-b)^{m-1}(a-x)^{n-1}\, dx = (a-b)^{m+n-1} B(m, n)$$

1.3.3 Relation between Beta and Gamma Function

To prove that $B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$, $m > 0, n > 0$

Proof: We know that

$$\frac{\Gamma(n)}{z^n} = \int_0^\infty e^{-zy} y^{n-1} dy \qquad ...(1)$$

or,
$$\Gamma(n) = \int_0^\infty z^n e^{-zy} y^{n-1} dy \qquad ...(2)$$

Also,
$$\Gamma(m) = \int_0^\infty e^{-y} y^{m-1} dy \qquad ...(3)$$

Now multiply (2) by $z^{m-1}e^{-z}$ on both sides, we get

$$\Gamma(n) \,.\, e^{-z} \,.\, z^{m-1} = \int_0^\infty z^{m+n-1} e^{-zy} y^{n-1} e^{-z} dy$$

$$= \int_0^\infty z^{m+n-1} e^{-(y+1)z} y^{n-1}\, dy$$

Integrating both the sides w.r.t. z from 0 to ∞, we get

$$\Gamma(n) \int_0^\infty e^{-z} z^{m-1} dz = \int_0^\infty \left[\int_0^\infty z^{m+n-1} e^{-(y+1)z} dz \right] y^{n-1} dy$$

$$\Rightarrow \qquad \Gamma(n)\,\Gamma(m) = \int_0^\infty \frac{\Gamma(m+n)}{(y+1)^{m+n}} \,.\, y^{n-1} dy \qquad \text{[by the property (1) and (3)]}$$

$$\Gamma(m)\,\Gamma(n) = \Gamma(m+n) \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$= \Gamma(m+n) \,.\, B(m, n) \qquad \text{[By (1) of (}iii\text{) of 1.3.2]}$$

Thus, $$B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}, \; m > 0, n > 0$$

Deduction (*i*), $\Gamma(n)\,\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$ where $0 < n < 1$

Proof: We know that

$$\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = B(m, n) = \int_0^x \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

Choose $m + n = 1$, so that $m = (1 - n)$

$$\therefore \qquad \frac{\Gamma(1-n)\Gamma(n)}{\Gamma(1)} = \int_0^\infty \frac{y^{n-1}}{1+y} dy, \; 0 < n < 1$$

$$\Rightarrow \qquad \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} \qquad \left[\because \; \Gamma(1) = 1, \int_0^\infty \frac{y^{n-1}}{1+y} dy = \frac{\pi}{\sin n\pi} \right]$$

where $0 < n < 1$

Deduction (*ii*) To prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof: Since $\Gamma(n)\,\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$

Put $n = \dfrac{1}{2}$, we obtain

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{\pi}{\sin\frac{\pi}{2}}$$

$$\Rightarrow \qquad \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi$$

$$\Rightarrow \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Deduction (*iii*) To prove that

$$\int_0^{\pi/2} \cos^m \theta \sin^n \theta \, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}, \; m > -1, n > -1$$

Proof: Let $I = \int_0^{\pi/2} \cos^m \theta \sin^n \theta \, d\theta$

Put $\sin^2 \theta = x$

$$\Rightarrow \quad 2 \sin \theta \cos \theta \, d\theta = dx$$

$$\Rightarrow \quad d\theta = \frac{dx}{2\sqrt{x} \, . \, \sqrt{1-x}}$$

Also, when $\theta = 0$, $x = 0$ and when $\theta = \frac{\pi}{2}$, $x = 1$

$$\therefore \quad I = \int_0^1 (1-x)^{\frac{m}{2}} \, . \, x^{\frac{n}{2}} \frac{dx}{2\sqrt{x} \, . \, \sqrt{(1-x)}}$$

$$= \frac{1}{2}\int_0^1 x^{\frac{n-1}{2}} \, . \, (1-x)^{\frac{m-1}{2}} \, dx$$

$$= \frac{1}{2}\int_0^1 x^{\frac{n+1}{2}-1} (1-x)^{\frac{m+1}{2}-1} \, dx$$

$$= \frac{1}{2} B\left(\frac{n+1}{2}, \frac{m+1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)} \qquad \left[\because \; B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$$

SOME SOLVED EXAMPLES

Example 1.52. *Evaluate the integral* $\int_0^1 x^4 (1-\sqrt{x})^5 \, dx$.

Solution. Let $\sqrt{x} = y \;\Rightarrow\; x = y^2 \;\Rightarrow\; dx = 2y \, dy$

$\therefore$ given integral becomes

$$\int_0^1 y^8 (1-y)^5 \, . \, 2y dy = 2\int_0^1 y^{(10-1)} (1-y)^{6-1} \, dy$$

$$= 2B(10, 6) = 2\frac{\Gamma(10)\Gamma(6)}{\Gamma(16)} = \frac{2.9!5!}{15!} = \frac{1}{15015} \quad \textbf{Answer}$$

Example 1.53. *Evaluate* $\int_0^1 (1-x^3)^{-1/2}\,dx$

Solution. Let $x^3 = y \quad\Rightarrow\quad x = y^{1/3} \quad\Rightarrow\quad dx = \frac{1}{3}y^{-2/3}\,dy$

$\therefore$ given integral becomes

$$\int_0^1 (1-x^3)^{-1/2}\,dx = \int_0^1 (1-y)^{-1/2}\cdot\frac{1}{3}y^{-2/3}\,dy$$

$$= \frac{1}{3}\int_0^1 y^{\frac{1}{3}-1}(1-y)^{\frac{1}{2}-1}\,dy$$

$$= \frac{1}{3}B\left(\frac{1}{3},\frac{1}{2}\right) = \frac{1}{3}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{6}\right)} \quad \textbf{Answer}$$

Example 1.54. *Express* $\int_0^1 y^m (1-y^p)^n\,dy$ *in terms of Beta function and hence evaluate* $\int_0^1 y^5 (1-y^3)^{10}\,dy$

Solution. Let $y^p = z \quad\Rightarrow\quad y = z^{1/p} \quad\Rightarrow\quad dy = \frac{1}{p}z^{\frac{1}{p}-1}\,dz$

$$\therefore \qquad \int_0^1 y^m(1-y^p)^n\,dy = \int_0^1 z^{\frac{m}{p}}(1-z)^n\cdot\frac{1}{p}z^{\frac{1}{p}-1}\,dz$$

$$= \frac{1}{p}\int_0^1 z^{\frac{m+1}{p}-1}(1-z)^{\overline{n+1}-1}\,dz = \frac{1}{p}B\left(\frac{m+1}{p}, n+1\right) \qquad ...(1)$$

Putting $m = 5, p = 3, n = 10$ in (1), we obtain

$$\int_0^1 y^5(1-y^3)^{10}\,dy = \frac{1}{3}B(2,11)$$

$$= \frac{1}{3}\frac{\Gamma(2)\,\Gamma(11)}{\Gamma(13)} = \frac{1}{13}\frac{1.\Gamma(1)\,\Gamma(11)}{12.11.\Gamma(11)}$$

$$= \frac{1}{3\times12\times11} \qquad [\because \quad \Gamma(n+1) = n\,\Gamma(n)]$$

$$= \frac{1}{396} \quad \textbf{Answer}$$

Example 1.55. *Evaluate* $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx$.

Solution. Let $$I = \int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx = \int_0^\infty \frac{x^8}{(1+x)^{24}}\,dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\,dx$$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\,dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\,dx$$

$$= B(9, 15) - B(15, 9)$$

$$= 0 \quad \textbf{Answer} \qquad [\because \quad B(m,n) = B(n,m)]$$

Example 1.56. *Prove that* $\int_0^{\pi/2}\left(1-\frac{\sin^2\theta}{2}\right)^{\frac{-1}{2}} d\theta = \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{4\sqrt{\pi}}$.

Solution. Let

$$I = \int_0^{\pi/2}\left(1-\frac{\sin^2\theta}{2}\right)^{-1/2} d\theta = \sqrt{2}\int_0^{\pi/2}\frac{d\theta}{\sqrt{2-\sin^2\theta}}$$

$$= \sqrt{2}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1+\cos^2\theta}}$$

Put $\cos\theta = t \Rightarrow -\sin\theta\, d\theta = dt \Rightarrow d\theta = -\frac{dt}{\sqrt{1-t^2}}$

$\therefore$

$$I = -\sqrt{2}\int_1^0\frac{dt}{\sqrt{1+t^2}\,.\,\sqrt{1-t^2}} = \sqrt{2}\int_0^1\frac{dt}{\sqrt{1-t^4}}$$

Again put $t^2 = \sin\theta \Rightarrow 2t\,dt = \cos\theta\,d\theta$

$\therefore$

$$I = \sqrt{2}\int_0^{\pi/2}\frac{\cos\theta\,d\theta}{2\sqrt{\sin\theta}\,.\,\sqrt{1-\sin^2\theta}} = \frac{1}{\sqrt{2}}\int_0^{\pi/2}\sin^{-1/2}\theta\cos^0\theta\,d\theta$$

$$= \frac{1}{2\sqrt{2}}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{1}{2\sqrt{2}}\frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2\pi}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} \quad \text{...(1)}$$

Since

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{\pi}{\sin\frac{\pi}{4}} = \pi\sqrt{2}$$

$\therefore$ (1) gives,

$$I = \frac{1}{2\sqrt{2}}\frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2\sqrt{\pi}}{\pi\sqrt{2}} = \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{4\sqrt{\pi}} \quad \textbf{Proved}$$

Example 1.57. *Evaluate* $\int_0^{\infty}\frac{x\,dx}{1+x^6}$.

Solution. Let

$$I = \int_0^{\infty}\frac{x\,dx}{1+x^6}$$

Putting $x^6 = y \Rightarrow x = y^{1/6} \Rightarrow dx = \frac{1}{6}y^{-5/6}\,dy$

$\therefore$

$$I = \int_0^{\infty}\frac{y^{1/6}}{(1+y)}\frac{1}{6}y^{-5/6}\,dy = \frac{1}{6}\int_0^{\infty}\frac{y^{-2/3}}{(1+y)}\,dy$$

$$= \frac{1}{6}\int_0^{\infty}\frac{y^{\frac{1}{3}-1}}{(1+y)^{\frac{1}{3}+\frac{2}{3}}}\,dy = \frac{1}{6}B\left(\frac{1}{3},\frac{2}{3}\right)$$

$$= \frac{1}{6}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)} = \frac{1}{6}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)}{\Gamma(1)}$$

$$= \frac{1}{6}\frac{\pi}{\sin\frac{\pi}{3}} \qquad \left[\because \ \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi} \text{ and } \Gamma(1) = 1\right]$$

$$= \frac{\pi}{3\sqrt{3}}.$$

Example 1.58. *Prove that* $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi.$

Solution. We know that

$$\int_0^{\pi/2} \sin^n\theta \cos^0\theta\, d\theta = \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{n+2}{2}\right)} \qquad \text{...(1)}$$

Now, let

$$I = \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}}$$

$$= \int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\, d\theta \times \int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta\, d\theta$$

$$= \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{5}{4}\right)} \times \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{4}\right)} \qquad \text{[By (1)]}$$

$$= \frac{\pi\Gamma\left(\frac{1}{4}\right)}{4\Gamma\left(\frac{5}{4}\right)} = \frac{\pi\Gamma\left(\frac{1}{4}\right)}{4\cdot\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} = \pi \quad \textbf{Proved} \qquad [\because \ \Gamma(n+1) = n\,\Gamma(n)]$$

Example 1.59. *Show that* $\int_0^{\infty} \frac{x^2 dx}{(1+x^4)^3} = \frac{5\pi\sqrt{2}}{128}.$

Solution. Put

$$x = \sqrt{\tan\theta} \quad \Rightarrow \quad dx = \frac{1}{2\sqrt{\tan\theta}} \cdot \sec^2\theta\, d\theta$$

$$\therefore \qquad \int_0^{\infty} \frac{x^2 dx}{(1+x^4)^3} = \int_0^{\pi/2} \frac{\tan\theta \cdot \frac{1}{2}(\tan\theta)^{-1/2}\sec^2\theta}{(1+\tan^2\theta)^3}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} (\tan\theta)^{1/2}(\sec\theta)^{-4}\, d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta\cos^{7/2}\theta\, d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{9}{4}\right)}{\Gamma(3)}$$

$$= \frac{1}{4.2!}\Gamma\left(\frac{3}{4}\right).\frac{5}{4}.\frac{1}{4}\Gamma\left(\frac{1}{4}\right)$$

$$= \frac{5}{128}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{5}{128}\,\frac{\pi}{\sin\frac{\pi}{4}} = \frac{5\pi\sqrt{2}}{128} \qquad \textbf{Proved}$$

Example 1.60. *Evaluate* $\int_0^\infty e^{-x^{1/3}}\,dx$.

Solution. Let $x^{1/3} = y \quad \Rightarrow \quad x = y^3 \quad \Rightarrow \quad dx = 3y^2\,dy$

$$\therefore \qquad \int_0^\infty e^{-x^{1/3}}dx = \int_0^\infty e^{-y}\,.3y^2dy$$

$$= 3\int_0^\infty e^{-y}y^{3-1}dy = 3\Gamma(3) = 3\,.\,2\,! = 6 \qquad \textbf{Answer}$$

1.3.4 Duplication Formula

To show that $\Gamma(p)\,\Gamma\left(p+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2p-1}}\,\Gamma(2p)$ where $p > 0$

Proof: We know that

$$\int_0^{\pi/2}\sin^m\theta\cos^n\theta\,d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)} \qquad ...(1)$$

Now putting $n = m$ in (1), we obtain

$$\frac{\left[\Gamma\left(\frac{m+1}{2}\right)\right]^2}{2\Gamma(m+1)} = \int_0^{\pi/2}(\sin\theta\cos\theta)^m\,d\theta$$

$$= \frac{1}{2^m}\int_0^{\pi/2}(2\sin\theta\cos\theta)^m\,d\theta = \frac{1}{2^m}\int_0^{\pi/2}(\sin 2\theta)^m\,d\theta$$

Again putting $2\theta = \phi \quad \Rightarrow \quad d\theta = \frac{d\phi}{2}$, we get

$$\frac{\left[\Gamma\left(\frac{m+1}{2}\right)\right]^2}{2\Gamma(m+1)} = \frac{1}{2^m}\int_0^{\pi}\sin^m\phi\,.\frac{d\phi}{2}$$

$$= \frac{1}{2^m}\int_0^{\pi/2}\sin^m\phi\cos^0\phi\,d\phi \qquad \left\{\int_0^{2a}f(x)\,dx = 2\int_0^a f(x)\,dx \text{ if } f(2a-x) = f(x)\right\}$$

$$= \frac{1}{2^m}\,\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{m+2}{2}\right)} \qquad \text{[by (1)]}$$

$$\Rightarrow \qquad \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma(m+1)} = \frac{\sqrt{\pi}}{2^m} \frac{1}{\Gamma\left(\frac{m+2}{2}\right)}$$

Let $\frac{m+1}{2} = p \Rightarrow m = 2p - 1, p > 0$

$$\therefore \qquad \frac{\Gamma(p)}{\Gamma(2p)} = \frac{\sqrt{\pi}}{2^{2p-1}} \cdot \frac{1}{\Gamma\left(\frac{2p+1}{2}\right)}$$

$$\Rightarrow \qquad \Gamma(p)\,\Gamma\left(p+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2p-1}}\,.\,\Gamma(2p) \text{ where } p > 0$$

which is known as Duplication formula

Deduction (*i*) To prove that

$$2^m\,\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{m+2}{2}\right) = \sqrt{\pi}\;\Gamma(m+1)$$

for all real values of m

Proof: Put $2p - 1 = m$ in Duplication formula, we get

$$\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{m+2}{2}\right) = \frac{\sqrt{\pi}}{2^m}\Gamma(m+1)$$

$$\Rightarrow \qquad 2^m\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{m+2}{2}\right) = \sqrt{\pi}\,\Gamma(m+1) \qquad \textbf{(proved)}$$

Deduction (*ii*) To prove that

$$\Gamma\left(m+\frac{1}{2}\right) = \frac{(2m)!}{2^m\,.\,m!}\sqrt{\pi}\text{, where } m > 0$$

Proof: We have $\frac{\Gamma(2m)}{\Gamma(m)} = \frac{(2m-1)!}{(m-1)!} = \frac{2m\,.\,(2m-1)!}{2m\,.\,(m-1)!} = \frac{(2m)!}{2\,.\,m!}$ ...(1)

Now using Duplication formula, we have

$$\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\frac{\Gamma(2m)}{\Gamma(m)}$$

$$= \frac{\sqrt{\pi}}{2^{2m-1}} \cdot \frac{(2m)!}{2\,.\,(m)!} \qquad \text{[by (1)]}$$

$$= \frac{\sqrt{\pi}}{2^m}\frac{(2m)!}{m!} \qquad \textbf{Proved}$$

SOME SOLVED EXAMPLES

Example 1.61. *Prove that* $B(n, n) = 2^{1-2n} B\left(n, \frac{1}{2}\right)$.

Solution. We have $B\left(n,\frac{1}{2}\right) = \dfrac{\Gamma(n)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)} = \dfrac{\Gamma(n)\,\Gamma(n)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)}$

$$= \frac{\Gamma(n)\,\Gamma(n)\,\sqrt{\pi}}{\sqrt{\pi}\,\Gamma(2n)\,.\,2^{1-2n}} \qquad \text{[By Duplication formula]}$$

$$= \frac{B(n,n)}{2^{1-2n}}$$

$$\Rightarrow \qquad B(n,n) = 2^{1-2n}\,B\left(n,\frac{1}{2}\right)$$

Example 1.62. *Prove that* $\Gamma\left(\frac{1}{6}\right) = \dfrac{\sqrt{3}}{2^{1/3}\sqrt{\pi}}\left[\Gamma\left(\frac{1}{3}\right)\right]^2.$

Solution. By Duplication formula, we have

$$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m)$$

Putting $m = \frac{1}{6}$, we get

$$\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{2}{3}\right) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{3}\right)}{2^{-2/3}}$$

$$\therefore \qquad \Gamma\left(\frac{1}{6}\right) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{3}\right)}{2^{-2/3}\,\Gamma\left(\frac{2}{3}\right)} \qquad ...(1)$$

Again, we know that

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

Putting $n = \frac{1}{3}$, we obtain

$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{\pi}{\sin\frac{\pi}{3}} = \frac{2\pi}{\sqrt{3}}$$

$$\therefore \qquad \Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}\,\Gamma\left(\frac{1}{3}\right)} \qquad ...(2)$$

Using (2), (1) becomes

$$\Gamma\left(\frac{1}{6}\right) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{3}\right)}{2^{-2/3}}\,.\,\frac{\sqrt{3}\,\Gamma\left(\frac{1}{3}\right)}{2\pi} = \frac{\sqrt{3}}{2^{1/3}\sqrt{\pi}}\left[\Gamma\left(\frac{1}{3}\right)\right]^2 \qquad \textbf{Proved}$$

Example 1.63. *Show that* $\int_{-\infty}^{\infty} \cos\left(\frac{\pi}{2}x^2\right)dx = 1$ *using Beta-Gamma functions.*

Solution. Let $$I = \int_{-\infty}^{\infty} \cos\left(\frac{\pi}{2}x^2\right)dx = 2\int_0^{\infty} \cos\left(\frac{\pi}{2}x^2\right)dx \qquad ...(1)$$

Now substituting $$\frac{\pi x^2}{2} = y \quad \Rightarrow \quad x = \sqrt{\frac{2}{\pi}}\, y^{1/2}$$

$\therefore$ $$dx = \frac{1}{2}\left(\sqrt{\frac{2}{\pi}}\right) y^{-1/2}\, dy$$

$$I = 2 \cdot \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^{\infty} y^{-1/2} \cos y\, dy = \sqrt{\frac{2}{\pi}} \int_0^{\infty} y^{-1/2} \cos y\, dy$$

$$= \text{Real part of } \sqrt{\frac{2}{\pi}} \int_0^{\infty} y^{-1/2} e^{-iy}\, dy \qquad [\because \; e^{-i\theta} = \cos\theta - i\sin\theta]$$

$$= \text{R.P. of } \sqrt{\frac{2}{\pi}} \int_0^{\infty} y^{-1/2} e^{-iy}\, dy$$

$$= \text{R.P. of } \sqrt{\frac{2}{\pi}} \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{e^{i\pi/2}}} \qquad \left[\because \; \int_0^{\infty} e^{-kx} x^{n-1}\, dx = \frac{\Gamma(n)}{k^n}\right]$$

$$= \text{R.P. of } \sqrt{\frac{2}{\pi}} \frac{\sqrt{\pi}}{\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2}} \qquad [\because \; i = e^{i\pi/2}]$$

$$= \text{R.P. of } \sqrt{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{-1/2}$$

$$= \text{R.P. of } \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) \qquad \text{[by Demoivre's theorem]}$$

$$= \sqrt{2}\cos\frac{\pi}{4} = \sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1 \quad \textbf{Proved}$$

Example 1.64. *Calculate* $\int_0^{\infty} \cos(\lambda^2 x^2)\, dx$, *using Beta-Gamma functions.*

Solution. Let $$I = \int_0^{\infty} \cos(\lambda^2 x^2)\, dx$$

Putting $$x^2 = z$$

$\Rightarrow$ $$dx = \frac{1}{2} z^{\frac{-1}{2}}\, dz$$

$\therefore$ $$I = \int_0^{\infty} \cos\lambda^2 z \cdot \frac{1}{2} z^{\frac{-1}{2}}\, dz = \frac{1}{2}\int_0^{\infty} z^{\frac{1}{2}-1} \cos\lambda^2 z\, dz$$

$$= \text{R.P. of } \frac{1}{2}\int_0^\infty e^{i\lambda^2 z} z^{\frac{1}{2}-1}\, dz$$

$$= \text{R.P. of } \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)}{(i\lambda^2)^{1/2}}$$

$$= \text{R.P. of } \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)}{\lambda(e^{i\pi/2})^{1/2}} \qquad [(i) = e^{i\pi/2}]$$

$$= \text{R.P. of } \frac{\sqrt{\pi}}{2\lambda}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{-1/2} \qquad \left[\because \int_0^\infty e^{-kx}\,.\,x^{n-1}\,dx = \frac{\Gamma(n)}{k^n}\right]$$

$$= \text{R.P. of } \frac{\sqrt{\pi}}{2\lambda}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) \qquad [\because \text{ by Demoivre's theorem}]$$

$$= \frac{\sqrt{\pi}}{2\lambda}\cos\frac{\pi}{4} = \frac{1}{2\lambda}\sqrt{\frac{\pi}{2}} \quad \textbf{Proved}$$

Example 1.65. *Evaluate* $\int_0^1 \log\Gamma(y)\,dy$ *, using Beta-Gamma functions.*

Solution. Let $I = \int_0^1 \log\Gamma(y)\,dy$ ...(1)

or, $I = \int_0^1 \log\Gamma(1-y)\,dy$ ...(2) $\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

Adding (1) and (2), we get

$$2I = \int_0^1 [\log\Gamma(y) + \log\Gamma(1-y)]\,dy$$

$$= \int_0^1 \log[\Gamma(y)\,\Gamma(1-y)]\,dy$$

$$= \int_0^1 \log\left(\frac{\pi}{\sin\pi y}\right)dy \qquad \left[\because \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \int_0^1 (\log\pi - \log\sin\pi y)\,dy$$

$$= \log\pi\,[y]_0^1 - \int_0^\pi \log\sin z\,.\,\frac{1}{\pi}\,dz \qquad [\text{for 2}^{\text{nd}} \text{ integral, put } \pi y = z]$$

$$= \log\pi - \frac{2}{\pi}\int_0^{\pi/2} \log\sin z\,dz \qquad \left[\because \int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx \text{ if } f(a-x) = f(x)\right]$$

$$= \log\pi - \frac{2}{\pi}\left(-\frac{\pi}{2}\log 2\right) \qquad \left[\because \int_0^{\pi/2} \log\sin x\,dx = -\frac{\pi}{2}\log 2\right]$$

$$= \log\pi + \log 2 = \log 2\pi \qquad [\text{Refer Example 1.31}]$$

$$\therefore \qquad I = \frac{1}{2}\log 2\pi \quad \textbf{Answer}$$

Example 1.66. *Prove that* $\Gamma\left(\frac{3}{2}-m\right)\Gamma\left(\frac{3}{2}+m\right)=\left(\frac{1}{4}-m^3\right)$ *π sec πm, provided –1 < 2m < 1.*

Solution.

$$\begin{aligned}
\text{L.H.S.} &= \Gamma\left(\frac{3}{2}-m\right)\Gamma\left(\frac{3}{2}+m\right)\\
&= \left(\frac{1}{2}-m\right)\Gamma\left(\frac{1}{2}-m\right).\left(\frac{1}{2}+m\right)\Gamma\left(\frac{1}{2}+m\right)\\
&= \left(\frac{1}{4}-m^2\right)\Gamma\left(\frac{1}{2}-m\right)\Gamma\left(\frac{1}{2}+m\right) = \left(\frac{1}{4}-m^2\right)\Gamma\left(\frac{1-2m}{2}\right)\Gamma\left(1-\frac{1-2m}{2}\right)\\
&= \left(\frac{1}{4}-m^2\right)\frac{\pi}{\sin\left(\frac{1-2m}{2}\right)\pi} = \left(\frac{1}{4}-m^2\right)\frac{\pi}{\sin\left(\frac{\pi}{2}-\pi m\right)}\\
&= \left(\frac{1}{4}-m^2\right)\frac{\pi}{\cos \pi m} = \left(\frac{1}{4}-m^2\right)\pi \sec \pi m\\
&= \text{R.H.S.} \quad \textbf{Proved}
\end{aligned}$$

Example 1.67. *Show that* $\int_0^{\pi/2} \tan^n\theta\, d\theta = \frac{\pi}{2}\sec\frac{n\pi}{2}$, *–1 < n < 1.*

Solution. Let

$$\begin{aligned}
I &= \int_0^{\pi/2} \tan^n \theta\, d\theta = \int_0^{\pi/2} \sin^n \cos^{-n} \theta\, d\theta\\
&= \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1-n}{2}\right)}{2\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(1-\frac{n+1}{2}\right)\\
&= \frac{1}{2}\frac{\pi}{\sin\left(\frac{n+1}{2}\right)\pi} = \frac{1}{2}\frac{\pi}{\sin\left(\frac{\pi}{2}+\frac{n\pi}{2}\right)} = \frac{1}{2}\frac{\pi}{\cos\frac{n\pi}{2}}\\
&= \frac{1}{2}\sec\frac{n\pi}{2} \quad \textbf{Proved}
\end{aligned}$$

EXERCISE 1.6

1. Evaluate the following integrals:

a. $\int_0^\infty \sqrt{x}\, e^{-x^3} dx$

b. $\int_0^\infty (8-x^3)^{-1/3} dx$

c. $\int_0^\infty e^{-x^2} x^{-1/2} dx \int_0^\infty x^2 e^{-x^4} dx$

d. $\int_0^\infty x^6 e^{-2x} dx$

e. $\int_0^\infty \frac{e^{-pt}}{\sqrt{t}} dt$

f. $\int_0^\infty \frac{x\, dx}{1+x^6}$

g. $\int_0^3 \frac{dt}{\sqrt{3t-t^2}}$

h. $\int_0^1 \frac{dt}{\sqrt{-\log t}}$

2. Prove that $\int_0^1 t^m (\log t)^n \, dt = \frac{(-1)^n \, . \, n!}{(m+1)^{n+1}}$, where n is a positive integer and $m > -1$.

3. Show that $\int_0^\infty \frac{t^{m-1}}{(a+bt)^{m+n}} \, dt = \frac{B(m,n)}{a^n b^m}$, where m, n, a and b are positive integers.

4. Prove that $\int_0^\pi \sqrt{\frac{\sin\theta}{(5+3\cos\theta)^n}} \, d\theta = \frac{\left[\Gamma\left(\frac{3}{4}\right)\right]^2}{2\sqrt{2\pi}}$

5. Evaluate the integrals:

i. $\int_0^1 \frac{dx}{(1-x^n)^{1/n}}$ ii. $\int_0^1 x^m (1-x^m)^p \, dx$ iii. $\int_0^1 \frac{dt}{(1-t^n)^{1/2}}$ iv. $\int_0^\infty x^m e^{-ax^n} \, dx$

6. Evaluate i. $\Gamma\left(-\frac{5}{2}\right)$ ii. $\Gamma\left(-\frac{15}{2}\right)$

7. Evaluate the following integrals:

i. $\int_0^1 \sqrt{1-x^4} \, dx$ ii. $\int_0^1 \frac{x^{m-1}}{(1+ax)(1-x)^m} \, dx$ iii. $\int_0^{\pi/2} \sin^2 x \, dx \times \int_0^{\pi/2} \sin^{q+1} x \, dx$

8. Show that $1 \, . \, 3 \, . \, 5 \, . \, ... \, (2m-1) = \frac{2^m \Gamma\left(m+\frac{1}{2}\right)}{\sqrt{\pi}}$

9. Prove that $\int_0^{\pi/2} \sqrt{\cot\theta} \, d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$

10. Prove that $\int_0^{\pi/2} (\sqrt{\tan\theta} + \sqrt{\sec\theta}) \, d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) + \sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)$

Answers

1. a. $\frac{\sqrt{\pi}}{3}$ b. $\frac{2\pi}{3\sqrt{3}}$ c. $\frac{\pi}{4\sqrt{2}}$ d. $\frac{45}{8}$

e. $\sqrt{\frac{\pi}{p}}, \, p > 0$ f. $\frac{\pi}{3\sqrt{3}}$ g. π h. $\sqrt{\pi}$

5. i. $\frac{\pi}{n} \sin\frac{\pi}{n}$ ii. $\frac{\Gamma(p+1)\Gamma\left(\frac{m+1}{n}\right)}{n\Gamma\left(p+1+\frac{m+1}{n}\right)}$ iii. $\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{n}\right)}{\pi\,\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}$ iv. $\frac{1}{n\,a^{(m+1)/n}}\Gamma\left(\frac{m+1}{n}\right)$

6. i. $\frac{-8}{15}\sqrt{\pi}$ ii. $\frac{\sqrt{\pi}2^8}{1.3.5.7...15}$

7. i. $\frac{[\Gamma(1/4)^2]}{6\sqrt{2\pi}}$ ii. $\frac{1}{(1+a)^m} \cdot \frac{\pi}{\sin m\pi}$ iii. $\frac{\pi}{2(q+1)}$

INTERESTING FACT

From Feynman diagrams (which involves the pictorial representation of subatomic particles), to Maxwell-Boltzmann statistics and distribution (which used in physics, chemistry and statistical mechanics to determine speed of molecules), these functions includes some real ground applications.

VIDEO REFERENCES

Beta & Gamma Function

USES OF ICT

- https://youtu.be/9_m36W3cK74
- https://youtu.be/3Co68ALYRTI

APPLICATIONS TO REAL LIFE

- It has many applications in strong nuclear forces.
- Beta distribution is used, when we solve time management problems.
- Gamma function is used to find time-based occurrences, such as life span of anything.
- In packing problems like, a cube will fit better in a sphere or a sphere in a cube.

1.4 APPLICATIONS OF DEFINITE INTEGRALS TO EVALUATE SURFACE AREAS AND VOLUMES OF REVOLUTION

If a plane area is revolved about a fixed line in its own plane, then the body so generated by the plane area is called the volume of the solid of revolution and the surface so generated is called the surface of revolution and the fixed line about which the solid revolves is called the axis of revolution.

Examples:

1. When a right angled triangle is revolved about its hypotenuse, then the double cone is formed.
2. When a circle is rotated about its diameter, a sphere is generated.

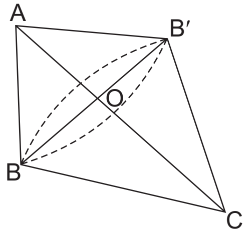


Fig. 1.17

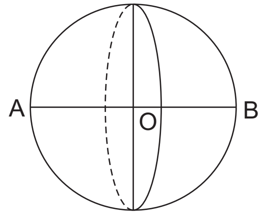


Fig. 1.18

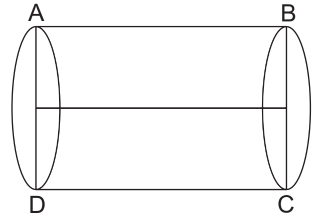


Fig. 1.19

3. When a square is rotated about any of its side, a right circular cylinder generated.

1.4.1 Volumes of Solids of Revolution

1.4.1.1 For Cartesian Curves

1. **Revolution about *x*-axis:** The volume of solid generated by the revolution about x-axis of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates, $x = a$, $x = b$ is $\int_a^b \pi y^2 dx$.
2. **Revolution about *y*-axis:** The volume of solid generated by the revolution about y-axis of the area bounded by the curve $x = f(y)$, the y-axis and the abscissas $y = a$, $y = b$ is $\int_a^b \pi y^2\, dx$.
3. **Revolution about any axis:** The volume of the solid generated by the revolution about any axis CD of the area bounded by the curve AB, the axis CD and the perpendiculars AC, BD on the axis is $\int_{OC}^{OD} \pi (PM)^2\, d(OM)$ where O is a fixed point on the axis CD and PM is perpendicular from any point P of the curve AB on CD.

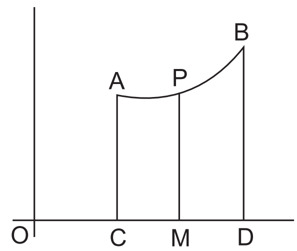


Fig. 1.20

1.4.1.2 For Parametric Curves

1. The volume of the solid generated by the revolution about x-axis, of the area bounded by the curve $x = f(t)$, $y = \phi(t)$, the x-axis and the ordinates at the points $t = a$, $t = b$ is $\int_a^b \pi x^2 \frac{dy}{dt} dt$.
2. The volume of the solid generated by the revolution about y-axis, of the area bounded by the curve $x = f(t)$, $y = \phi(t)$, the y-axis and the abscissas at the points $t = a$, $t = b$ is $\int_a^b \pi x^2 \frac{dy}{dt} dt$.
3. Volume of solid of revolution between two solids: The volume of the solid generated by the revolution about the x-axis of the area bounded by the curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $x = a$, $x = b$ is $\int_a^b \pi [f_1^2(x) - f_2^2(x)]\, dx$ where $f_1(x)$ is the ordinates of the upper curve and $f_2(x)$ is that of the lower curve.

1.4.1.3 For Polar Curves

The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$

1. about the initial line OX $(\theta = 0)$ is $\int_\alpha^\beta \frac{2}{3} \pi r^3 \sin\theta\, d\theta$
2. about the line OY $\left(\theta = \frac{\pi}{2}\right)$ is $\int_\alpha^\beta \frac{2}{3} \pi r^3 \cos\theta\, d\theta$

1.4.2 Surface Areas of Solid of Revolution

1.4.2.1 For Cartesian Curve

The curved surface of the solid generated by the revolution about x-axis, of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is $\int_a^b 2\pi y \frac{ds}{dx} dx$ where $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

1.4.2.2 For Parametric Curve

The curved surface of the solid generated by the revolution about x-axis, of the area bounded by the curve $x = f(t)$, $y = \phi(t)$, the x-axis and the ordinates at the points $t = a$, $t = b$ is

$$\int_a^b 2\pi y \frac{ds}{dt}\, dt \quad \text{where } \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

1.4.2.3 For Polar Curve

The curved surface of the solid generated by the revolution, about the initial line of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$ is

$$\int_{\theta=\alpha}^{\theta=\beta} 2\pi y \frac{ds}{d\theta}\, d\theta \quad \text{where } \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \text{ and } y = r\sin\theta.$$

SOME SOLVED EXAMPLES

Example 1.68. *Find the volume of the solid generated by revolving the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *about the major axis.*

Solution. The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

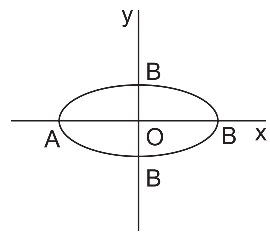


Fig. 1.21

The ellipse is symmetrical about y-axis.

Required volume of solid generated by the ellipse about x-axis

$$= 2\int_0^a \pi y^2\, dx = 2\pi \int_0^a \frac{b^2}{a^2}(a^2 - x^2)\, dx$$

$$= 2\pi \frac{b^2}{a^2}\int_0^a (a^2 - x^2)\, dx = 2\pi \frac{b^2}{a^2}\left|a^2 x - \frac{x^3}{3}\right|_0^a = \frac{4}{3}\pi ab^2$$

Example 1.69. *Find the volume generated by the revolution about the initial line of the limacon $r = a + b\cos\theta$, $a > b$.*

Solution. We shall revolve only the shaded region above the initial line

$$V = \frac{2}{3}\pi \int_0^{\pi} (a + b\cos\theta)^2 \sin\theta\, d\theta$$

$$a + b\cos\theta = t \quad b\sin\theta\, d\theta = -\,dt$$

$$= -\frac{2}{3}\frac{\pi}{b}\left[\frac{t^3}{3}\right]_{a+b}^{a-b} = -\frac{2\pi}{3b}\left[\frac{-6a^2b - 2b^3}{3}\right]$$

$$= \frac{4}{9}\pi(b^2 + 3a^2).$$

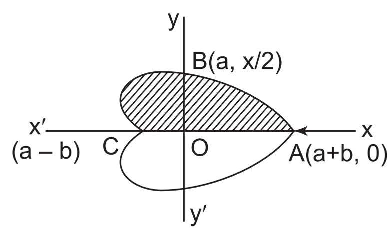


Fig. 1.22

Example 1.70. *Find the volume of the solid generated by the revolution of the loop of the curve $(a - x)\, y^2 = (a + x)\, x^2$ about the x-axis.*

Solution. The shape of the curve is shown in the figure.

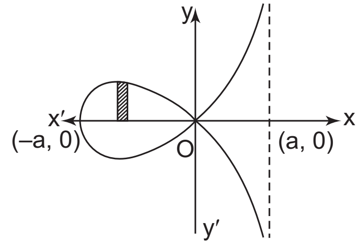


Fig. 1.23

The curve is symmetrical about x-axis

$\therefore$ Required volume, $V = \int_{-a}^{0} \pi y^2 dx = \int_{-a}^{0} \pi \frac{(a+x)}{(a-x)} x^2 dx$

Let $a - x = z,\ dx = -\,dz$

when $x = -a,\ z = 2a$

$x = 0,\ \ z = a$

$$V = \pi\int_{2a}^{a} \frac{(a+a-z)}{z} (a-z)^2\,(-dz) = \pi\int_{a}^{2a} \frac{2a-z}{z} (a-z)^2\, dz$$

$$= \pi\int_{a}^{2a} \frac{(2a-z)}{z} (a^2 + z^2 - 2z)\, dz$$

$$= \pi\int_{a}^{2a} \left[\frac{2a^3}{z} + 2az - 4a^2 - a^2 - z^2 + 2az\right] dz$$

$$= \pi\int_{a}^{2a} \left[\frac{2a^3}{z} + 4az - z^2 - 5a^2\right] dz = \pi\left[2a^3 \log z + 2az^2 - \frac{z^3}{3} - 5a^2 z\right]_a^{2a}$$

$$= \pi\left[\left\{2a^3 \log 2a + 2a\,(2a)^2 - \frac{(2a)^3}{3} - 5a^2\,(2a)\right\} - \left\{2a^3 \log a + 2a^3 - \frac{a^3}{3} - 5a^3\right\}\right]$$

$$= 2\pi a^3\left[\log 2 - \frac{2}{3}\right]$$

Example 1.71. *Find the volume generated by the revolution of the area under one complete arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, the axis of revolution being the x-axis.*

Solution. Let the area under arc be divided into n strips of width dx by lines || to the y-axis and y be the height of the typical strip at a distance x from y-axis.

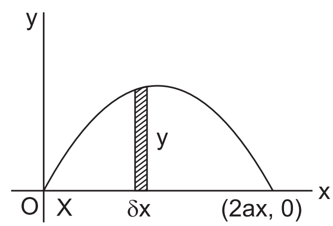


Fig. 1.24

$$\text{Volume} = \int_0^{2\pi} \pi y^2 dx = \int_0^{2\pi} \pi a^2\,(1 - \cos\theta)^2\, a\,(1 - \cos\theta)\, d\theta$$

$$= \int_0^{2\pi} \pi a^3\,(1 - \cos\theta)^3\, d\theta$$

$$= \int_0^{2\pi} \pi a^3 \times \left(2\sin^2\frac{\theta}{2}\right)^3 d\theta$$

$$= 8\pi a^3 \int_0^{2\pi} \sin^6\left(\frac{\theta}{2}\right) d\theta = 16\pi a^3 \int_0^{\pi} \sin^6\phi\, d\phi \qquad (\theta = 2\phi)$$

$$= 32\pi a^3 \int_0^{\pi/2} \sin^6\phi\, d\phi \qquad (\sin(\pi - \phi) = \sin\phi)$$

$$= 32\pi a^3\, \frac{5\times3\times1}{6\times4\times2}\, \frac{\pi}{2} = 5\pi^2 a^3. \qquad (\because \text{ By using Reduction formula})$$

Example 1.72. *Find the volume of the surface generated by revolving the Cardioid $r = a(1 + \cos\theta)$ about initial line.*

Solution. Volume generated

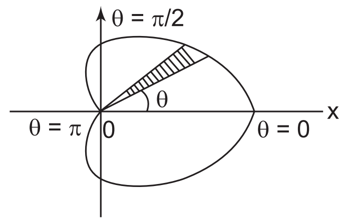


Fig. 1.25

$$= \frac{2}{3}\pi\int_0^{\pi} r^3 \sin\theta\, d\theta$$

$$= \frac{2}{3}\pi\int_0^{\pi} a^3 (1+\cos\theta)^3 \sin\theta\, d\theta$$

$$= -\frac{2}{3}\pi a^3 \int_0^{\pi}(1+\cos\theta)^3(-\sin\theta)\, d\theta$$

$$= \frac{-2\pi}{3}a^3\left[\frac{(1+\cos\theta)^4}{4}\right]_0^{\pi} = \frac{2}{3}\pi a^3\left[\frac{(1+1)^4}{4}\right] = \frac{8}{3}\pi a^3$$

Example 1.73. *Show that the volume of the solid formed by the revolution of the Cissoid $y^2(2a - x) = x^3$ about its asymptotes is $2\pi^2a^3$.*

Solution. The asymptote of this curve is $x = 2a$.

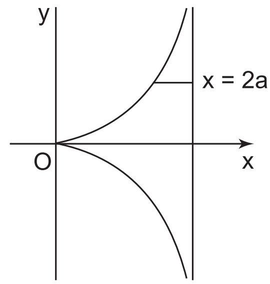


Fig. 1.26

Required volume $\qquad V = 2\pi\int_0^{2a}(2a-x)^2\, dy$

From the equation of the curve

$$y = \frac{x^{3/2}}{\sqrt{2a-x}}$$

$$\frac{dy}{dx} = \frac{(3a-x)\sqrt{x}}{(2a-x)^{3/2}}$$

$$dy = \frac{(3a-x)\sqrt{x}}{(2a-x)^{3/2}}\, dx$$

When y varies from 0 to ∞, x varies from 0 to $2a$

$$V = 2\pi\int_0^{2a}(2a-x)^2\, dy = 2\pi\int_0^{2a}(2a-x)^2\,\frac{(3a-x)\sqrt{x}}{(2a-x)^{3/2}}\, dx$$

$$= 2\pi\int_0^{2a}(a-x)\sqrt{2ax-x^2}\, dx + 4\pi a\int_0^{2a}\sqrt{2ax-x^2}\, dx$$

Put $x = 2a\sin^2\theta$, we get

$$\int_0^{2a}\sqrt{2ax-x^2}\, dx = \int_0^{\pi/2} 2a\times 4a\sin^2\theta\cos^2\theta\, d\theta = \pi\frac{a^2}{2}$$

$$V = \pi\left[\frac{(2ax-x^2)^{3/2}}{3/2}\right]_0^{2a} + 4\pi a\,\frac{\pi a^2}{2} = 0 + 2\pi^2 a^3 = 2\pi^2 a^3$$

Example 1.74. *The area cut off from the right parabola $y^2 = 4ax$ by the chord joining the vertex to an end of the latus rectum through four right-angles about the chord. Find the volume of the solid generated.*

Solution. The equation of the parabola is

$$y^2 = 4ax$$

The co-ordinates of B are $(a, 2a)$

Hence the equation of *OB*

$$y - 0 = \frac{2a-0}{a-0}(x-0)$$

$$2x - y = 0$$

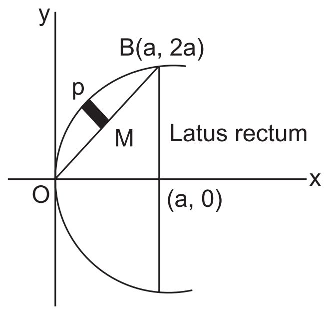


Fig. 1.27

Let $P(at^2, 2at)$ be a point on the arc *OB* and *PM* the perpendicular from *P* to *OB*.

$$PM = \frac{2at^2 - 2at}{\sqrt{4+1}} = \frac{2at\,(t-1)}{\sqrt{5}}$$

$$OP = \sqrt{(at^2-0)^2 + (2at-0)^2} = at\sqrt{t^2+4}$$

$$OM^2 = OP^2 - PM^2 = a^2\,t^2(t^2+4) - \frac{4a^2t^2\,(t-1)^2}{5}$$

$$OM^2 = \frac{a^2t^2(t+4)^2}{5}$$

$$OM = \frac{at\,(t+4)}{\sqrt{5}}$$

$$\text{Required volume} = \int_{t=0}^{t}\pi(PM)^2\,d(OM) = \pi\int_0^1 \frac{4a^2t^2(t-1)^2}{5}\,d\left[\frac{at\,(t+4)}{\sqrt{5}}\right]$$

$$= \frac{4\pi a^3}{5\sqrt{5}}\int_0^1 t^2(t^2-2t+1)(2t+4)\,dt = \frac{4\pi a^3}{5\sqrt{5}}\int_0^1 (2t^5-6t^3+4t^2)dt = \frac{2\pi a^3}{15\sqrt{5}}$$

Example 1.75. *Find the surface area of the solid generated by the revolution of the ellipse $x^2 + 4y^2 = 16$ about the major axis.*

Solution. The equation of the ellipse is

$$x^2 + 4y^2 = 16$$

$$4y^2 = 16 - x^2 \quad \therefore \quad y = \frac{\sqrt{16-x^2}}{2}$$

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{2}\frac{1}{\sqrt{16-x^2}}\times(-2x) = -\frac{x}{2\sqrt{16-x^2}}$$

$$dy = -\frac{x}{2\sqrt{16-x^2}}\,dx$$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x^2}{4(16-x^2)}} = \sqrt{\frac{64-3x^2}{4(16-x^2)}}$$

The ellipse $x^2 + 4y^2 = 16$ meets *x*-axis where $y = 0$,

$$x^2 = 16 \quad x = \pm 4$$

For the upper half of the ellipse in first quadrant *x* varies from 0 to 4.

The ellipse is symmetrical about *y*-axis

$$\therefore \quad \text{Required surface} = 2\times\int_0^4 2\pi\,y\,\frac{ds}{dx}\,dx = 4\pi\int_0^4 \frac{\sqrt{16-x^2}}{2}\times\sqrt{\frac{64-3x^2}{4(16-x^2)}}\,dx$$

$$= \pi \int_0^4 \sqrt{64 - 3x^2}\, dx = \sqrt{3}\,\pi \int_0^4 \sqrt{\frac{64}{3} - x^2}\, dx$$

$$= \sqrt{3}\pi \left[\frac{x\sqrt{\frac{64}{3} - x^2}}{2} + \frac{64}{3 \times 2} \sin^{-1} \frac{x}{8} \right]_0^4$$

$$= \sqrt{3}\,\pi \left[2\sqrt{\frac{64}{3} - 16} + \frac{32}{3} \sin^{-1} \frac{\sqrt{3}}{2} \right]$$

$$= \sqrt{3}\,\pi \left[2 \times \frac{4}{\sqrt{3}} + \frac{32}{3}\frac{\pi}{3} \right] = 8\pi \left[1 + \frac{4\pi}{3\sqrt{3}} \right]$$

Example 1.76. *Find the area of the surface of revolution generated by revolving one arc of the curve $y = \sin x$ about the x-axis.*

Solution. One arc of the curve $y = \sin x$ lies in $(0, \pi)$. Further

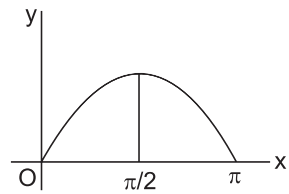


Fig. 1.28

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \cos^2 x}$$

$$\text{Required surface area} = 2\pi \int_0^{\pi} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_0^{\pi} \sin x \sqrt{1 + \cos^2 x}\, dx$$ Put $t = \cos x \quad \therefore \quad dt = -\sin x\, dx$

$$= -2\pi \int_1^{-1} \sqrt{1 + t^2}\, dt = 2\pi \int_{-1}^{1} \sqrt{1 + t^2}\, dt = 2\pi \left[\frac{t\sqrt{1 + t^2}}{2} + \frac{1}{2} \sin h^{-1}(1) \right]_{-1}^{1}$$

$$= \pi \{ [\sqrt{2} + \sinh^{-1}(1)] - [-\sqrt{2} - \sinh^{-1}(1)] \} = 2\pi [\sqrt{2} + \sinh^{-1}(1)]$$

Example 1.77. *Find the surface of the solid generated by revolving the astroid $x = a \cos^3 t$, $y = a \sin^3 t$ about the axis of x.*

Solution. Let the shaded portion OAB be revolved above x-axis.

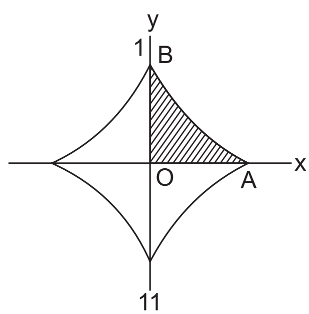


Fig. 1.29

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(-3a \cos^2 t \sin t)^2 + (3a \sin^2 t \cos t)^2}$$

$$= \sqrt{9a^2 \cos^2 t \sin^2 t \times (\cos^2 t + \sin^2 t)}$$

$$= 3a \cos t \sin t.$$

$$\text{Surface area,} \quad S = 2\int_0^{\pi/2} 2\pi\, y \frac{ds}{dt} dt = 4\pi \int_0^{\pi/2} a \sin^3 t \times 3a \cos t \sin t\, dt$$

$$= 12\pi a^2 \int_0^{\pi/2} \sin^4 t \cos t\, dt = 12\pi a^2 \left[\frac{\sin^5 t}{5} \right]_0^{\pi/2} = \frac{12}{5} \pi a^2$$

Example 1.78. *A quadrant of a circle of radius 'a' bounded by the tangents at its extremities revolves about one of the tangents. Show that the surfaces area so generated is* $\pi(\pi - 2)a^2$.

Solution. Let the equation of the circle be

$$x^2 + y^2 = a^2 \text{ or } x = a\cos t, y = a \sin t$$

and $P(x, y)$ be any point on it

$$PM = NA = a - x = a - a\cos t = a(1 - \cos t)$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} = a$$

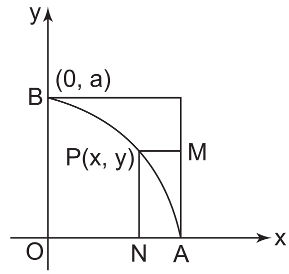


Fig. 1.30

Surface area, $S = \int_0^{\pi/2} 2\pi (PM)^2 \times \frac{ds}{dt} \times dt$

$$= 2\pi a^2 \int_0^{\pi/2} (1 - \cos t)\, dt$$

$$= 2\pi a^2 \int_0^{\pi/2} (1 - \cos t)\, dt = 2\pi a^2 [t - \sin t]_0^{\pi/2}$$

$$= 2\pi a^2 \left[\frac{\pi}{2} - 1\right]$$

$$= \pi a^2(\pi - 2)$$

Example 1.79. *Find the surface of the solid formed by revolving the cardioid* $r = a(1 + \cos\theta)$ *about the initial line.*

Solution. The cardioid is symmetrical about the initial line and for its upper half, θ varies from 0 to π.

Also $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2 (1 + \cos\theta)^2 + a^2 \sin^2\theta}$

$$= a\sqrt{2(1 + \cos\theta)} = a\sqrt{4\cos^2 \theta/2} = 2a\cos\frac{\theta}{2}$$

$\therefore$ Required surface $= \int_0^{\pi} 2\pi y \frac{ds}{d\theta} d\theta = 2\pi \int_0^{\pi} r\sin\theta \times 2a\cos\frac{\theta}{2} d\theta$ $\quad \left\{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right.$

$$= 4\pi a \int_0^{\pi} a(1 + \cos\theta)\sin\theta \cos\frac{\theta}{2} d\theta$$

$$= 4\pi a^2 \int_0^{\pi} 2\cos^2\frac{\theta}{2} \times 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{\theta}{2} d\theta$$

$$= 16\pi a^2 \int_0^{\pi} \cos^4\frac{\theta}{2} \sin\frac{\theta}{2} d\theta$$

$$= 16\pi a^2 (-2) \int_0^{\pi} \cos^4\frac{\theta}{2}\left(-\sin\frac{\theta}{2} \times \frac{1}{2}\right) d\theta$$

$$= -32\pi a^2 \left[\frac{\cos^5 \theta/2}{5}\right]_0^{\pi} = -\frac{32\pi a^2}{5}(0 - 1) = \frac{32\pi a^2}{5}$$

Example 1.80. *Find the surface of the sphere of radius 'a', equation of the circle being r = a.*

Solution. Equation of the circle is $r = a$. Let the shaded portion be revolved about x-axis.

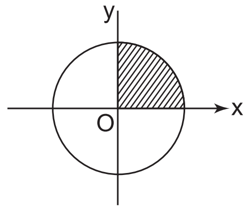


Fig. 1.31

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2 + 0} = a$$

$$S = 2\int_0^{\pi/2} 2\pi r \sin\theta \frac{ds}{d\theta}\, d\theta$$

$$= 4\pi \int_0^{\pi/2} a \sin\theta \times a\, d\theta$$

$$= 4\pi \int_0^{\pi/2} a^2 \sin\theta\, d\theta$$

$$= 4\pi a^2 [-\cos\theta]_0^{\pi/2}$$

$$= 4\pi a^2.$$

EXERCISE 1.7

1. Find the surface area of the solid generated by the revolution of astroid $x^{2/3} + y^{2/3}$ about x-axis.
2. Find the surface area of the solid generated by revolving the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about x-axis.
3. Determine the area of the surface generated by the revolution of the loops of the curve $r^2 = a^2 \cos 2\theta$ about initial line.
4. Find the volume of the solid formed by the revolution of the curve $y^2(a + x) = x^2(a - x)$ about x-axis.
5. The arc lying between $\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$ of the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ rotates about the axis of x. Find the volume of the solid so generated.
6. Find the volume generated by revolving the area bounded by the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$, $x = 0$, $y = 0$ about x-axis.
7. Find the volume formed by the revolution of the curve $27ay^2 = 4(x - 3a)^2$ about x-axis.
8. Area bounded by x-axis, $y^2 = 4ax$ and the ordinate $x = 3a$ is revolved about x-axis. Find the volume generated.
9. Show that the volume of the solid generated by revolving the area included between the curves $y^2 = x^3$, $x^2 = y^3$ about x-axis is $\frac{5\pi}{28}$.
10. Find the surface and volume of the ellipsoid formed by the revolution of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its major axis respectively.
11. The part of the parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the volume of the reel thus generated.
12. The part of the parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the area of the surface so generated.
13. For the curve $r^2 = a^2 \cos 2\theta$, prove that the volume of revolution about the initial line is $\frac{\pi a^2}{6\sqrt{2}}[3\log(\sqrt{2}+1) - \sqrt{2}]$

14. Find the volume of the solid generated by revolving the area included between the curve $\frac{y+8}{x} = x - 2$ and x-axis about the line $x + 5 = 0$.

15. Find the surface area of the solid formed by the revolution of the loop of the curve given by $3ay^2 = x(x - a)^2$.

16. Find the volume and surface of the solid generated by the revolution of the loop of the curve $x = t^2$, $y = \frac{t^3}{3}$ about x-axis.

17. A quadrant of a circle of radius 'a' revolves about its chord. Show that the volume of the spindle generated is $\frac{\pi a^3}{6\sqrt{2}}(10 - 3\pi)$.

Answers

1. $\frac{12}{5}\pi a^2$ **2.** $\frac{64}{3}\pi a^2$ **3.** $2\pi a^2 (2 - \sqrt{2})$ **4.** $2\pi a^3 \left[\log 2 - \frac{2}{3}\right]$

5. $\frac{16\pi a^2}{105}$ **6.** $\frac{\pi a^3}{12}$ **7.** $48\pi a^3$ **8.** $18\pi a^3$

10. $2\pi ab = \sqrt{1 - e^2} + e^{-1}\sin^{-1} e$ and $\frac{4}{3}\pi ab^2$ **11.** $\frac{4}{5}\pi a^3$

12. $\pi a^2 [3\sqrt{2} + \log(\sqrt{2} - 1)]$ **14.** 432π **15.** $\frac{4a}{\sqrt{3}}$

16. $\frac{3\pi}{4}, 3\pi$

INTERESTING FACTS

- Do you know, in electrical circuits, there exists a relationship between current and charge which can be calculated by this concept. (https://www.math24.net/integrals-electric-circuits)
- Engineering work in various industries utilizes the knowledge of this concept to find the Centre of Mass and Moment of Inertia of any object.
- Architects, while constructing any building use this concept.

VIDEO REFERENCES

Application of Definite Integral - I

APPLICATIONS TO REAL LIFE

- This concept is used in business and economics domain to calculate "*Lorenz curve and Gini coefficient*", and increase the total profit.
- Application in physics to find the mass and density of any object.
- It is also used to find average changes, volumes, error estimations and surface areas.

SUBJECTIVE SOLVED QUESTIONS
(HOTS)

Example 1. *Evaluate* $\int_{-3/2}^{10} \{2x\}\, dx$, *where* {.} *denotes the fractional part of x.*

Solution. We know that $f(x) = \{2x\}$ is a periodic function with period $\frac{1}{2}$

Let $$I = \int_{-3/2}^{10} \{2x\}\, dx = \int_{-3(1/2)}^{20(1/2)} \{2x\}\, dx$$

$$= 23\int_{0}^{1/2} 2x\, dx \qquad (\text{as } \{2x\} = 2x - [2x] \text{ and when } x \in [0, 1/2], [2x] = 0)$$

$$= \left|23x^2\right|_0^{1/2}$$

$$= \frac{23}{4}$$

Remark. If $f(x)$ is a periodic function with period p, then $\int_{a/np}^{b/np} f(x)dx = \int_a^b f(x)dx, n \in I$.

Example 2. *Evaluate* $\int_{-1}^{1} x^3 \,.\, e^{x^4}\, dx.$

Solution. Let $f(x) = x^3 e^{x^4}$, then

$$f(-x) = (-x)^3 \,.\, e^{(-x)^4} = -x^3 e^{x^4} = -f(x)$$

Hence $f(x)$ is an odd function.

$$\therefore \qquad \int_{-1}^{1} f(x)dx = 0; \text{ or } \int_{-1}^{1} x^3 e^{x^4} dx = 0$$

Example 3. *Evaluate* $\int_0^1 \frac{dx}{(1-x^n)^{1/n}}$

Solution. Let $$I = \int_0^1 \frac{dx}{(1-x^n)^{1/n}}$$

Put $x^n = \sin^2\theta$ *i.e.* $x = \sin^{2/n}\theta$

$$dx = \frac{2}{n}\sin^{\left(\frac{2}{n}-1\right)}\theta \cos\theta d\theta$$

So, $$I = \frac{2}{n}\int_0^{\pi/2} \frac{\sin^{\left(\frac{2}{n}-1\right)}\theta\cos\theta\,d\theta}{\cos\theta}$$

$$= \frac{2}{n}\int_0^{\pi/2} \sin^{\frac{2}{n}-1}\theta\,d\theta$$

$$= \frac{2}{n}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$$

$$= \frac{\sqrt{\pi}}{n}\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{n+2}{2n}\right)}$$

Example 4. *Evaluate* $\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}}$.

Solution. Let
$$x = a\cos^2\theta + b\sin^2\theta$$
$$dx = 2a\cos\theta\sin\theta d\theta + 2b\sin\theta\cos\theta d\theta$$
$$= 2(b-a)\sin\theta\cos\theta d\theta$$
$$x - a = a\cos^2\theta + b\sin^2\theta - a$$
$$= (b-a)\sin^2\theta$$
$$b - x = b - a\cos^2\theta - b\sin^2\theta$$
$$= (b-a)\cos^2\theta$$

So, $$I = \int_0^{\pi/2}\frac{2(b-a)\sin\theta\cos\theta}{(b-a)\sin\theta\cos\theta}d\theta$$
$$= 2\int_0^{\pi/2} d\theta = \pi$$

Example 5. *Show, by means of a suitable substitution, that*

$$\int_0^{\pi/2}\sin^{2x-1}\theta\cos^{2y-1}\theta d\theta = \frac{1}{2}\int_0^{\infty}\frac{t^{x-1}}{(1+t)^{x+y}}dt,\ x, y > 0$$

Solution. Let
$$\sin\theta = \frac{1}{\sqrt{1+z}}$$
$$\cos\theta d\theta = \frac{-1}{2}(1+z)^{-3/2}\,dz$$
$$\cos\theta = \frac{2^{1/2}}{\sqrt{1+z}}$$
$$I = \int_0^{\pi/2}\sin^{2x-1}\cos^{2y-1}\theta d\theta$$

$$= -\frac{1}{2}\int_{\infty}^{0}\frac{1}{(1+z)^{x-1/2}}\cdot\frac{z^{y-1}}{(1+z)^{y-1}}\cdot\frac{1}{(1+z)^{3/2}}dz$$

$$= \frac{1}{2}\int_{0}^{\infty}\frac{z^{y-1}}{(1+z)^{x+y}}\,dz = \frac{1}{2}B(y, x)$$

Since, $B(x, y) = B(y, x)$

So, $I = \frac{1}{2}\int_{0}^{\infty}\frac{z^{y-1}}{(1+z)^{x+y}}\,dz$

Example 6. *If* $I_n = \int_0^{\pi/2} sin^n\, x\, dx$, *then show that* $I_n = \left(\frac{n-1}{n}\right)I_{n-2}$.

Solution. Given $I_n = \int_0^{\pi/2} \sin^n x\, dx$

$$I_n = \left[-\sin^{n-1} x\cos x\right]_0^{\pi/2} + \int_0^{\pi/2} (n-1)\sin^{n-2} x\,.\cos^2 x\, dx$$

$$= (n-1)\int_0^{\pi/2} \sin^{n-2} x\,(1-\sin^2 x)dx$$

$$= (n-1)\int_0^{\pi/2} \sin^{n-2} x\, dx - (n-1)\int_0^{\pi/2} \sin^n x\, dx$$

$$I_n + (n-1)\, I_n = (n-1)\, I_{n-2}$$

$$I_n = \left(\frac{n-1}{n}\right)I_{n-2}$$

Example 7. *The part of the parabola* $y^2 = 4ax$ *cut off by the latus rectum revolves about the tangent at the vertex. Find the curved surface of the reel thus generated.*

Solution. The given parabola is $y^2 = 4ax$

Differentiating (1) w.r.t. x, we get $dy/dx = 2a/y$

$$\frac{ds}{dx} = \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left\{1+\frac{4a^2}{y^2}\right\}}$$

$$= \sqrt{\left\{1+\frac{4a^2}{4ax}\right\}} = \sqrt{\left(\frac{x+a}{x}\right)}$$

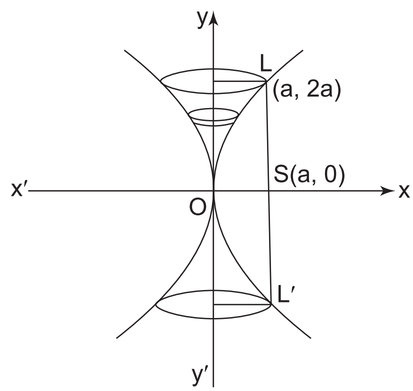


Fig. 1.32

The required curved surface is generated by the revolution of the arc LOL' (LSL' is the latus rectum), about the tangent at the vertex *i.e.*, y-axis. The curve is symmetrical about x-axis and for the arc OL, x varies from 0 to a.

$\therefore$ The required surface $S = 2\int_0^a 2\pi x\frac{ds}{dx}dx$

$$= 4\pi\int_0^a x\sqrt{\left(\frac{x+a}{x}\right)}dx = 4\pi\int_0^a \sqrt{(x^2+ax)}dx$$

$$= 4\pi\int_0^a \sqrt{\left\{\left(x+\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2\right\}}dx$$

$$= 4\pi\left[\frac{1}{2}\left(x+\frac{a}{2}\right)\sqrt{(x^2+ax)} - \frac{1}{2}.\frac{a^2}{4}\log\left\{\left(x+\frac{a}{2}\right)+\sqrt{(x^2+ax)}\right\}\right]_0^a$$

$$\left[\because \int\sqrt{(x^2-a^2)}dx = \frac{1}{2}x\sqrt{(x^2-a^2)} - \frac{1}{2}a^2\log\left\{x+\sqrt{(x^2-a^2)}\right\}\right]$$

$$= 4\pi\left[\frac{1}{2}.\frac{3}{2}aa\sqrt{2} - \frac{1}{8}a^2\log\left\{\frac{3}{2}a+a\sqrt{2}\right\} + \frac{1}{2}a^2\log\left(\frac{1}{2}a\right)\right]$$

$$= 4\pi\left[\frac{3}{4}a^2\sqrt{2} - \frac{1}{8}a^2\log\left\{\left(\frac{3}{2}a+a\sqrt{2}\right)/\left(\frac{1}{2}a\right)\right\}\right]$$

$$= \pi a^2\left[3\sqrt{2} - \frac{1}{2}\log\left(3+2\sqrt{2}\right)\right]$$

$$= \pi a^2\left[3\sqrt{2} - \frac{1}{2}\log\left(\sqrt{2}+1\right)^2\right]$$

$$= \pi a^2\left[3\sqrt{2} - \log\left(\sqrt{2}+1\right)\right]$$

Example 8. *Find the area of the surface of the solid bounded by the cone* $z = 3 - \sqrt{x^2+y^2}$ *and the Paraboloid* $z = 1 + x^2 + y^2$.

Solution. Hint. Convert in polar coordinated then solve.

Example 9. *The part of the ellipse* $x^2/a^2 + y^2/b^2 = 1$ *cut off by a latus rectum revolves about the tangent at the nearer vertex. Find the volume of the reel thus generated.*

Solution. The given ellipse is $x^3/a^2 + y^2/b^2 = 1$. The focus of ellipse is $(ae, 0)$ where e is the eccentricity given by $e = \sqrt{1-\frac{b^2}{a^2}}$. The line segment passing through focus and intercepted by ellipse is called latus rectum.

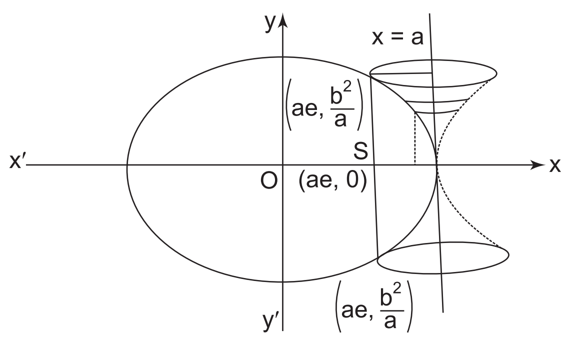


Fig. 1.33

Consider a volume element in the form of disc of radius $(a - x)$ and thickness dy. The volume of this element $\pi(a-x)^2\,dy$.

The volume of solid of revolution is given by

$$V = \int_{-b^2/a}^{b^2/a} \pi(a-x)^2\,dy$$

$$= 2\int_0^{b^2/a} \pi(a-x)^2\,dy \qquad \left[\text{where } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ i.e. } x^2 = \frac{a^2}{b^2}(b^2-y^2)\right]$$

$$= 2\pi\int_0^{b^2/a} (a^2 - 2ax + x^2)\,dy$$

$$= 2\pi\int_0^{b^2/a} \left\{a^2 - 2b\,.\,\frac{a}{b}\sqrt{(b^2-y^2)} + \frac{a^2}{b^2}(b^2-y^2)\right\}dy$$

$$= \frac{2\pi a^2}{b^2}\int_0^{b^2/a} \left\{2b^2 - 2b\sqrt{(b^2-y^2)} - y^2\right\}dy$$

$$= \frac{2\pi a^2}{b^2}\left[2b^2 y - 2b\left\{\frac{2}{2}y\sqrt{(b^2-y^2)} + \frac{1}{2}b^2\sin^{-1}\left(\frac{y}{b}\right)\right\} - \frac{y^3}{3}\right]_0^{b^2/a}$$

$$= \frac{2\pi a^2}{b^2}\left[2b^2\,.\,\frac{b^2}{a} - 2b\left\{\frac{1}{2}\frac{b^2}{a}\,.\sqrt{\left(b^2 - \frac{b^4}{a^2}\right)} + \frac{1}{2}b^2\sin^{-1}\frac{b}{a}\right\} - \frac{b^6}{3a^3}\right]$$

$$= \frac{2\pi a^2}{b^2}\left[2\frac{b^4}{a} - \frac{b^4}{a^2}\,.\sqrt{(a^2-b^2)} - b^3\sin^{-1}\frac{b}{a} - \frac{b^6}{3a^3}\right]$$

$$= \frac{2\pi b}{3a}\left\{6a^2 b - 3ab\sqrt{(a^2-b^2)} - 3a^3\sin^{-1}\frac{b}{a} - b^3\right\}$$

Example 10. *Find the volume of the solid generated by the revolution of the curve* $y = \dfrac{a^3}{(a^2+x^2)}$ *about its asymptote.*

Solution. Answer: $\frac{1}{2}\pi^2 a^3$

Hint:

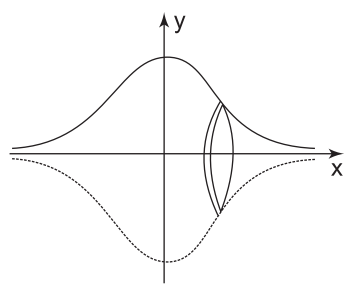


Fig. 1.34

Example 11. *Show that the volume of the solid generated by the revolution of the curve* $(a-x)y^2 = a^2x$, *about its asymptote is* $\frac{1}{2}\pi^2 a^3$.

Solution. The given curve is $(a-x)y^2 = a^2x$. Its shape is as shown in the figure. Equating to zero, the coefficient of highest power of y, the asymptote parallel to the axis of y is $a - x = 0$ *i.e.*, $x = a$.

Consider a volume element in the form of disc of radius $(a - x)$ and thickness dy. The volume of this element $d\tau = \pi(a-x)^2\, dy$.

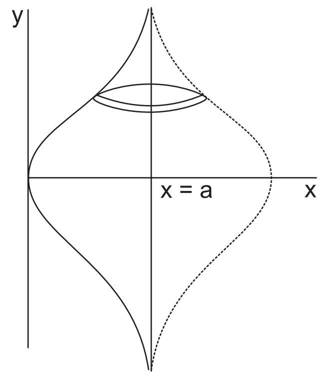


Fig. 1.35

$\therefore$ The required volume

$$V = 2\int_{y=0}^{\infty} \pi(a-x)^2\, dy$$

$$= 2\pi\int_0^{\infty}\left(a - \frac{ay^2}{y^2+a^2}\right)^2 dy \qquad \left[\because \text{ from (1), } x = \frac{ay^2}{y^2+a^2}\right]$$

$$= 2\pi a^6 \int_0^{\infty} \frac{dy}{(y^2+a^2)^2}$$

Now, put $y = a\tan\theta$ so that $dy = a\sec^2\theta d\theta$. When $y = 0$, $\theta = 0$ and when $y \to \infty$, $\theta \to \pi/2$.

Therefore, the required volume

$$= 2\pi a^6 \int_0^{\pi/2} \frac{a\sec^2\theta d\theta}{a^4\sec^4\theta} = 2\pi a^3 \int_0^{\pi/2} \cos^2 d\theta$$

$$= 2\pi a^3 . \frac{1}{2} . \frac{1}{2}\pi = \frac{1}{2}\pi^2 a^3$$

Example 12. *Discuss the convergence of the Beta function.* [*M.D.U. 2012; K.U.*]

Or

Show that $\int_0^1 x^{m-1}(1-x)^{n-1}\, dx$ *exists if and only if m, n are both positive.*

[*K.U. 2012, 08; M.D.U. 2008*]

Solution. Let $$I = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

The integral I is proper if $m \geq 1$ and $n \geq 1$ and so it is convergent of $m \geq 1$ and $n \geq 1$ Clearly 0 and 1 are the points of infinite discontinuity if $m < 1$ and $n < 1$ respectively. For $m < 1$ and $n < 1$, take a number $\frac{1}{2}$(say) between 0 and 1, so that we can write

$$I = \int_0^{1/2} x^{m-1}(1-x)^{n-1}\, dx + \int_{1/2}^1 x^{m-1}(1-x)^{n-1}\, dx$$

Let, $$I = I_1 + I_2 \qquad ...(1)$$

To discuss the convergence of $I_1 = \int_0^{1/2} x^{m-1}(1-x)^{n-1}\, dx$ *at* $x = 0$ *when* $m < 1$:

Here $$f(x) = x^{m-1}(1-x)^{n-1} = \frac{(1-x)^{n-1}}{x^{1-m}} \qquad [\because \ m < 1]$$

Take $g(x) = \frac{1}{x^{1-m}}$ so that $\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+}(1-x)^{n-1} = 1$, which is finite and non-zero

$\therefore$ I_1 and $\int_0^{1/2} g(x)dx = \int_0^{1/2} \frac{1}{x^{1-m}} dx$ converge or diverge together.

But $\int_0^{1/2} \frac{1}{x^{1-m}} dx$ converges at $x = 0$ iff $1 - m < 1$ i.e., $m > 0$.

$\therefore$ I_1 converges at 0 iff $0 < m < 1$.

To discuss the convergence of $I_2 = \int_{1/2}^{1} x^{m-1}(1-x)^{n-1}dx$ *at* $x = 1$ *when* $n < 1$:

Here $$f(x) = x^{m-1}(1-x)^{n-1} = \frac{x^{m-1}}{(1-x)^{1-n}} \qquad [\because \quad n < 1]$$

Take $g(x) = \frac{1}{(1-x)^{1-n}}$ so that $\lim_{x \to 1^-} \frac{f(x)}{g(x)} = \lim_{x \to 1^-} x^{m-1} = 1$, which is finite and non-zero

$\therefore$ I_2 and $\int_{1/2}^{1} g(x)\,dx$ converge or diverge together.

But $\int_{1/2}^{1} g(x)\,dx = \int_{1/2}^{1} \frac{1}{(1-x)^{n-1}} dx$ converges at $x = 1$ iff $1 - n < 1$, i.e., $n > 0$

$\therefore$ I_2 converges at 1 iff $0 < n < 1$.

Therefore from (1), the integral I is convergent if and only if $0 < m < 1$ and $0 < n < 1$. Also it is a proper integral for $m \geq 1$ and $n \geq 1$.

Hence I is convergent iff m. n are both positive.

Example 13. *Discuss the convergence of Gamma function.* [*M.D.U. 2013, 11, 01*]

Or

Show that the integral $\int_0^{\infty} x^{n-1} e^{-x}\, dx$ *is convergent if* $n > 0$. [*M.D.U. 2012, 07*]

Solution. Let $$I = \int_0^{\infty} x^{n-1} e^{-x}\, dx$$
$$= \int_0^{1} x^{n-1} e^{-x}\, dx + \int_1^{\infty} x^{n-1} e^{-x}\, dx$$

Let $$I = I_1 + I_2 \qquad ...(1)$$

Here $$f(x) = x^{n-1} e^{-x} = \frac{e^{-x}}{x^{1-n}}$$

The integrand f has an infinite discontinuity at $x = 0$ in $[0, 1]$ if $n < 1$ and I_1 is a proper integral and hence convergent for $n \geq 1$.

To test the convergence of I_1 *at 0 when* $n < 1$:

Take $g(x) = \frac{1}{x^{1-n}}$ so that $\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} e^{-x} = 1$, which is finite and non-zero.

But the integral $\int_0^1 g\,dx = \int_0^1 \frac{1}{x^{1-n}} dx$ is convergent if and only if $1 - n < 1$, i.e., $n > 0$.

Therefore by comparison test, the integral $\int_0^1 f\,dx$ converges for $0 < n < 1$. Also it is a proper integral for $n \geq 1$. Hence I_1 is convergent for all $n > 0$.

To test the convergence of I_2:

Take $g(x) = \dfrac{1}{x^2}$ so that $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \dfrac{x^{n+1}}{e^x} = 0$ for all x.

Now $\int_1^\infty g dx = \int_1^\infty \dfrac{1}{x^2} dx$ is convergent $[\because \quad n = 2 > 1]$

$\therefore$ By comparison test, the integral I_2 is also convergent for all n.

Hence by (1), we conclude that the given integral I is convergent if and only if $n > 0$.

SUMMARY

1. Radius of curvature $\rho = \dfrac{(1+y_1^{\,2})^{3/2}}{y_2}$.

2. Coordinates of the centre of curvature $\bar{x} = x - \rho \sin \psi$, $\bar{y} = y + \rho \sin \psi$

$\Rightarrow \quad \bar{x} = x - \dfrac{y_1}{y_2}(1+y_1^{\,2})$ and $\bar{y} = y + \dfrac{1}{y_2}(1+y_1^{\,2})$.

3. Evolute and Involute: The locus of the centre of curvature of a curve is called the evolute and the curve itself is called the involute.

4. If $f(x)$ is defined in the interval $[a, b]$, then the definite integral of $f(x)$ is written as

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

5. The integral $\int_a^b f(x)\,dx$ $f(x)$ is an improper integral, if either 'a' or 'b' or both 'a' and 'b' are infinite or the function $f(x)$ is unbounded on $[a, b]$.

6. Beta and Gamma functions

a. $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ converges for $m, n > 0$

b. $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$ converges for $n > 0$

c. $\Gamma(n+1) = n\,\Gamma(n)$ and $\Gamma(n-1) = n\,!$, if n is a positive integer

d. $\Gamma(1) = 1 = \Gamma(2)$, $\Gamma(1/2) = \Gamma(\pi)$

e. $\beta(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

f. $$\int_0^{\pi/2} \sin^p x \cos^q x\,dx = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$$

g. $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta\, d\theta = \dfrac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma(1)} = \dfrac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$

7. Surface area of solids of revolution

$$S = \int 2\pi y\, ds = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{(revolution about } x\text{-axis)}$$

$$S = \int 2\pi x\, ds = \int_a^b 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dy \qquad \text{(revolution about } y\text{-axis)}$$

Volume of solids of revolution

Revolution about x-axis $= \int_a^b \pi y^2\, dx$

Revolution about y-axis $= \int_c^d \pi x^2\, dy$

OBJECTIVE QUESTIONS

1. Find the value of $\int_0^{\pi/2} \sin^3 x \cos^{5/2} x\, dx$

a. $\frac{1}{77}$ b. $\frac{2}{77}$ c. $\frac{4}{77}$ d. $\frac{8}{77}$

2. The name of the evolute of an ellipse is

a. centroid b. astroid c. cycloid d. hyperboloid

3. Involute is also known as

a. evolute b. evolvent c. envelope d. tangent

4. What is the curvature of the curve $x^2 + y^2 = 25$?

a. 5 b. 25 c. 0.5 d. 0.2

5. What is the curvature of a straight line?

a. infinite b. one

c. zero d. length of the straight line

6. The value of the integral $\int_0^{\pi/2} [\tan^{-1}(\cot x) + \cot^{-1}(\tan x)]\, dx$ is

a. $\frac{\pi}{4}$ b. π c. $\frac{\pi^2}{4}$ d. $\frac{\pi^2}{2}$

7. If $I = \int_{-1}^{1} (x^7 + \cos^{-1} x)\, dx$, then $\cos I$ is equal to

a. 1 b. 0 c. –1 d. 1/2

8. The value of $\int_0^{\pi/2} \sin\theta \sqrt{\sin 2\theta}\, d\theta$ is ......

a. 1 b. 0 c. $\pi/2$ d. $\pi/4$

9. Which of the following is not a definition of Gamma function?

a. $\Gamma(n) = n\,!$ b. $\Gamma(n+1) = n\Gamma(n)$

c. $\Gamma(n) = \int_0^\infty x^{n-1}\, x^{-x}\, dx$ d. $\Gamma(n) = \int_0^1 \log\left(\frac{1}{y}\right)^{n-1}$

10. What is the value of $\int_0^1 \frac{x^2}{\sqrt{(1-x^4)}}\, dx$ is ....

a. $\frac{2\sqrt{\pi}\,\Gamma(5/4)}{\Gamma(1/4)}$ b. $\frac{2\pi\,\Gamma(3/4)}{\Gamma(1/4)}$ c. $\frac{2\sqrt{\pi}\,\Gamma(3/4)}{\Gamma(1/4)}$ d. $\frac{2\sqrt{\pi}\,\Gamma(3/4)}{\Gamma(5/4)}$

11. What is the value of $\Gamma(9/4)$?

a. $\frac{5}{4}\times\frac{1}{4}\times\Gamma\left(\frac{1}{4}\right)$ b. $\frac{9}{4}\times\frac{5}{4}\times\frac{1}{4}\times\Gamma\left(\frac{1}{4}\right)$ c. $\frac{5}{4}\times\frac{1}{4}\times\Gamma\left(\frac{5}{4}\right)$ d. $\frac{1}{4}\times\Gamma\left(\frac{1}{4}\right)$

12. What is the value of $\Gamma(n)\,\Gamma(1-n)$?

a. $\frac{\pi}{\sin n\pi}$ b. $\frac{-\pi}{\sin n\pi}$ c. 0 d. $n\,!$

13. How much volume generated when the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is revolved about its minor axis?

a. $4ab$ cubic units b. $\frac{4}{3}a^2b$ cubic units c. $\frac{4}{3}ab$ cubic units d. 4 cubic units

14. How much volume generated when the region surrounded by $y = \sqrt{x}$, $y = 2$ and $y = 0$ is revolved about y-axis?

a. 32π cubic units b. $\frac{32\pi}{5}$ cubic units c. $\frac{32}{5}$ cubic units d. $\frac{5\pi}{32}$ cubic units

15. What is the area of the cardioid $y = a(1 + \cos\theta)$?

a. $\frac{3}{2}\pi a^2$ b. $3\pi a^2$ c. $\frac{3}{4}\pi a^2$ d. $\frac{3}{8}\pi a^2$

Answers

1. d	**2.** b	**3.** b	**4.** d
5. c	**6.** c	**7.** c	**8.** d
9. a	**10.** c	**11.** a	**12.** a
13. b	**14.** b	**15.** a	

SUBJECTIVE UNSOLVED QUESTIONS
(HOTS)

1. Define the osculating plane of curve at a point and from this definition, find its equation.

2. Show that the locus of the centre of curvature is an evolute, only when the curve is a plane.

3. Prove that the distance between corresponding points of two involutes is constant.

4. Evaluate $\int_{-2}^{3} |x^2 - 1|\, dx$.

5. Discuss the convergence of the integral $\int_0^1 x^{n-1} \log x \, dx$.

6. Prove that the area of the surface $z^2 = 2xy$ included between the planes $x = 0, x = a, y = 0, y = a$ is $4\sqrt{ab}\,\dfrac{a+b}{3\sqrt{2}}$.

7. Find the area of the surface $az = xy$ that lies inside the cylinder$(x^2 + y^2)^2 = 2a^2 xy$.

8. Find the volume of the solid formed by revolving the cycloid about its base.

9. A quadrant of a circle of radius 'a' revolves about its chord. Find the volume of the spindle generated.

Answers

4. 28/3   **6.** convergent if $n > 0$ divergent if $n <= 0$

8. $1/9(20 - 3\pi)a^2$ **9.** $5\pi^2 a^3$ **10.** $\dfrac{\pi a^3(10 - 3\pi)}{6\sqrt{2}}$

DID YOU KNOW?

Newton described his version of differential calculus as 'the method of fluxions'. He wrote a paper on fluxions in 1666, but like many of his works, it was not published until decades later. His magnum opus *Philosophiae naturalis principia mathematica* (Mathematical principles of natural philosophy) was published in 1687. This work includes his theories of motion and gravitation, but does not include much calculus explicitly — although there is some explanation of calculus at the beginning, and Newton certainly used calculus to formulate his theories. Nonetheless, Newton's 'method of fluxions' did not explicitly appear in print until 1693.

Leibniz, on the other hand, published his first paper on calculus in 1684 — and claimed to have discovered calculus in the 1670s. From the published record, at least, Leibniz seemed to have discovered calculus first.

While Newton and Leibniz initially had a cordial relationship, Leibniz and his followers did not take kindly to a statement made by the English mathematician John Wallis. With a rather xenophobic and quarrelsome character, Wallis fought priority disputes on behalf of English scientists throughout his life. In 1695, perhaps inadvertently, Wallis intimated that Leibniz learned about calculus from Newton — a claim now known to be false.

Then, offended by a statement of Leibniz that certain mathematical problems could only be solved by Leibniz's own version of the calculus, a mathematician named Fatio de Duiller in 1699 accused Leibniz of plagiarism. Things only went downhill from there. It did not help matters that Newton and Leibniz also disagreed on philosophical questions.

In 1712 the Royal Society in England wrote a report purporting to settle the matter — except, the whole investigation was effectively directed by Newton himself. The report found that Leibniz had concealed his knowledge of Newton's work — based on facts now known to be false. In response, Leibniz accused Newton and his followers of stealing Leibniz's own calculus and making errors in their applications of it. The dispute went on well after Leibniz's death in 1716, full of accusations and counter-accusations.

Nobody came out of the dispute well. Both Newton and Leibniz were capable of incredible mathematical discoveries, but their dispute demonstrated they were also capable of some rather less impressive behaviour.

PROJECT/PRACTICAL/ACTIVITIES

PROJECT

1. Create your script for computing the envelope of a rational family of lines to compute the equation of the evolute of the following curves:

i. the parabola $y = x^2$ parameterized as $x(t) = t$, $y(t) = t^2$.

ii. The ellipse $x^2 + 4y^2 = 4$ parameterized as $x(t)= \frac{8t}{1+4t^2}$, $y(t) = \frac{4t^2-1}{1+4t^2}$.

Plot these evolutes along with their corresponding curves .

2. Relate Beta function and String theory.

PRACTICAL

1. Plot 3-D image of Beta Function.

2. Sketch a graph and shade the area of the specified range for $\int_1^4 (x+6)\,dx$.

3. Use definite integral, find the shaded area for the given curves:

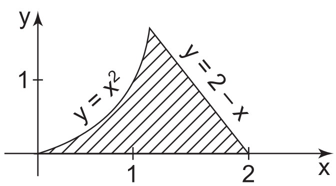
a. 

Fig. 1.36

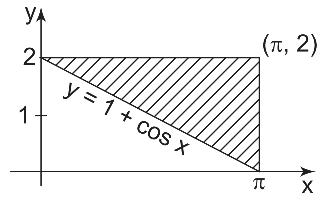
b. 

Fig. 1.37

4. Write a MATLAB Code to draw the graph of the evolute of a parabola.

ACTIVITY

i. How can the Gamma function interpolate the factorial function? (think and explain)

ii. How does the graph of Gamma function look like?
[**Hint:** Can use Python code.]

KNOW MORE

1. If 'f' is an even function and $\int_0^2 f(x)\,dx = k$, then $\int_{-1}^{1}\left(\frac{x^2-1}{x^2}\right) f\left(x+\frac{1}{x}\right)dx$ is equal to ......

a. 0 b. $2k$ c. k d. $4k$

2. The value of the integral $\int_{-5}^{5}(x-[x])\,dx$ is

a. 0 b. 5 c. 10 d. 15

3. $\int_0^2 x^{[x]}\,dx$ is equal to ......

a. $\frac{1}{2}$ b. $\frac{3}{2}$ c. $\frac{5}{2}$ d. $\frac{7}{2}$

4. Find the value of $\Gamma(0.1)\,\Gamma(0.2)\,\Gamma(0.3)$ ...... $\Gamma(0.9)$?

a. $\frac{(2\pi)^{9/2}}{\sqrt{10}}$ b. $\frac{(\pi)^2}{\sqrt{10}}$ c. $\frac{1}{2}$ d. $\frac{1}{\sqrt{2}}$

5. Examine the convergence of integral $A=\int_1^2 \frac{x}{\sqrt{x-1}}\,dx$, $B=\int_0^{\pi}\frac{dx}{1+\cos x}$.

6. Test the convergence of integral $\int_0^4 \frac{dx}{x(4-x)}$.

7. Let $\alpha=\int_0^{\infty}\frac{1}{1+t^2}\,dt$. Which of the following is true?

a. $\frac{d\alpha}{dt}=\frac{1}{1+t^2}$ b. α is a rational number

c. log $(\alpha)=1$ d. sin $(\alpha)=1$

Answers

1. b **2.** b **3.** c **4.** a
5. A convergest to 8/3 and B diverges to $+\infty$ **6.** divergent **7.** d

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2 Calculus II

UNIT SPECIFICS

This unit discusses the topics Rolle's theorem, it's geometrical interpretation, mean value theorems with their geometrical interpretation, Taylor's and Maclaurin's theorems with remainders, indeterminate forms and L'Hospital's rule (all types.), maxima and minima in length. The applications of various topics are discussed thoroughly and many solved examples are included for proper understanding of the topic. Many figures have been included so as to make students visualize the topics.

RATIONALE

Theorems lie at the core of mathematics. Theorems are often described as being "trivial", or "difficult", or "deep", or even "beautiful". These subjective judgments vary not only from person to person, but also with time and culture: for example, Rolle's theorem is used for analyzing the graphs of a company's yearly performance. Mean value theorem are often applied with motion problems such as throwing a ball into the air or else. These can be used as a mathematical tool in solving other problems related to computations.

Taylor Series are very useful to evaluate an approximation of many hard to calculate expressions. We use the L'Hospital Rule to solve the limits, it also has many applications in the real world, specially in statistics, physics, and engineering.

Everything in this world is based on the concept of maxima and minima, every time, everyone calculates the maximum and minimum value of every data.

PRE-REQUISITES

1. Concept of Continuity and differentiability.
2. Knowledge of special type of function such as mod, increasing function, decreasing function, open interval, closed interval.
3. Evaluation of limit, also applying it.
4. Aware with expansion of some functions like $\sin x$, $\cos x$, $\log(1 \pm x)$, e^x etc.

UNIT OUTCOMES

After completion of this unit, students will be able to:

U2-01: Apply the various Mean value theorems to prove the properties of a function comprised with its derivatives.

U2-02: Determine the asymptotic behaviour of function $f(x)$ as $x \to \infty$ and evaluate the limit using L'Hospital Rule.

U2-03: Analyse the behaviour of the function using Maxima-Minima.

U2-04: Learn about the expansion of series for the algebraic and transcendental function with Taylor' and Maclaurin's Theorem.

MAPPING OF UNIT OUTCOMES WITH COURSE OUTCOMES

Unit 2 Outcomes	EXPECTED MAPPING WITH COURSE OUTCOMES (1- Weak Correlation; 2- Medium Correlation; 3- Strong Correlation)				
	CO-1	CO-2	CO-3	CO-4	CO-5
U2-01	1	2	–	–	–
U2-02	1	3	–	–	–
U2-03	–	3	–	–	1
U2-04	–	3	–	–	–

HISTORY

Earlier of the 17th century, a curve was generally described as a locus of points satisfying some geometric condition, and tangent lines were obtained through geometric construction. The relation between tangent lines to curves and the velocity of a moving particle was discovered in the late 1660s by Isaac Newton. Rolle's Theorem is a part, of the Mean Value Theorem. Bhaskara II (1114-1185), an Indian mathematician, is credited with being the first person to employ Rolle's Theorem and it was named after Michel Rolle (1652-1719), a French mathematician. The theorem was considered to be part of infinitesimal calculus and was not categorized under differential calculus until the 18th century. Lagrange provided a result only by using the first two conditions of Rolle's theorem. Hence it is called Lagrange's Mean-Value Theorem. Cauchy gave another mean value theorem in which he used two functions instead of one function as in the case of Rolle's theorem and Lagrange's Mean-Value Theorem, Lagrange's theorem is a particular of Cauchy Mean Value Theorem. Taylor's Theorem can be regarded as an extension of the Mean Value Theorem to "higher order" derivatives. L'Hospital's Rule was in fact discovered by Johann Bernoulli. In 1955, the L'Hospital–Bernoulli correspondence was published in Germany.

—Bhaksara II (1114-1185)

2.1 ROLLE'S THEOREM

Statement: Let f be a function defined on $[a, b]$ be such that

i. f is continuous on $[a, b]$ ii. f is differentiable on (a, b) iii. $f(a) = f(b)$

then there exist at least one real number c lying between a and b such that $f'(c) = 0$

Proof: Since f is continuous on $[a, b] \Rightarrow f$ is bounded on $[a, b]$ and attains its bounds.

Let $\underset{x\in[a,b]}{\text{Sup.}} f(x) = M$ and $\underset{x\in[a,b]}{\text{Inf.}} f(x) = m$

By the property of continuity, there exists $c, d \in [a, b]$ such that

$f(c) = M$ and $f(d) = m$ [$\because$ if a function $f(x)$ is continuous on closed interval $[a, b]$, then it attains its supremum and infimum atleast once in $[a, b]$]

Two different cases arises:

Case I: When $M = m$ *i.e.* Sup. f = Inf. f

In this case, $f(x) = M(=m)$ for all $x \in [a, b]$

$\Rightarrow$ f is a constant function over $[a, b]$

$\therefore$ $f'(x) = 0$ for all $x \in [a, b]$

Hence $f'(c) = 0$ where $c \in (a, b)$

Case II: When $M \neq m$

Given that $f(a) = f(b)$

$\therefore$ either M or m is different from $f(a) = f(b)$

Suppose that $M \neq f(a)$ and $M \neq f(b)$

As $f(c) = M$, then $c \neq a$, $c \neq b$ and therefore $a < c < b$

Since $f(c) = M = \underset{x\in[a,b]}{\text{Sup.}} f(x)$

Therefore, $f(x) \leq f(c)$ for all $x \in [a, b]$ ...(1)

From (1) $f(c-h) \leq f(c)$

$\therefore$ $f(c-h) - f(c) \leq 0$

Dividing by $-h < 0$, we get

$$\frac{f(c-h)-f(c)}{-h} \geq 0$$

Taking limit as $h \to 0$

$$\lim_{h\to 0} \frac{f(c-h)-f(c)}{-h} \geq 0$$

$\Rightarrow$ $Lf'(c) \geq 0$ ...(2)

Again from (1), $f(c+h) \leq f(c)$

$\therefore$ $f(c+h) - f(c) \leq 0$

Dividing by $h > 0$, we get

$$\frac{f(c+h)-f(c)}{h} \leq 0$$

Taking limit as $h \to 0$

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \leq 0$$

$\Rightarrow$ $Rf'(c) \leq 0$ ...(3)

Since $f'(c)$ exists

$\therefore$ $Lf'(c) = Rf'(c) = f'(c)$

This is possible only when $f'(c) = 0$.

Proceeding in the same manner, we can prove that $f'(d) = 0$, where $d \in (a, b)$.

Hence there exist atleast one $c \in (a, b)$ such that $f'(c) = 0$.

This completes the proof of the theorem.

Remark: In the proof of Rolle's theorem, we used some terms (like Sup. Inf.). Here we are giving the brief explanation of those.

1. **Least upper bound (Supremum): Definition:** Let S be a non-empty subset of $\Re$. A real number u is said to be a least upper bound or (l.u.b.) or supremum of S if
   i. $x \le u \ \forall \ x \in S$ *i.e.*, u is an upper bound of S.
   ii. If v is an upper bound of S, then $u \le v$.
2. **Greatest lower bound (Infimum): Definition:** Let S be a non-empty subset of $\Re$. A real number l is said to be a greatest lower bound or (g.l.b.) or infimum of S if
   i. $l \le x \ \forall \ x \in S$ *i.e.*, l is a lower bound of S.
   ii. If l' is any lower bound of S, then $l' \le l$. In other words any number greater than l is not a lower bound of S.

For example:

1. If $S = (0, 1)$, then clearly $0 < x < 1 \ \forall \ x \in S$
   Also, $x < 2 \ \forall \ x \in S$
   $\therefore$ 2, 3, 4... and so on are upper bounds of S but 1 is the least upper bound among them.
   $\therefore$ 1 is l.u.b. of S and $1 \notin S$.
   Similarly, $-1 < x \ \forall \ x \in S$
   $\therefore$ $-1, -2, -3, \ldots$ and so on are lower bounds of S but among all these lower bound, 0 is greatest.
   $\therefore$ 0 is g.l.b. of S and $0 \notin S$.
2. If $S = [0, 1]$, then $0 \le x \le 1 \ \forall \ x \in S$
   Here, 1 is l.u.b. of S and $1 \in S$.
   Also, 0 is g.l.b. of S and $0 \in S$.

Important Points:

i. l.u.b. or g.l.b. of a set if exist is unique.
ii. l.u.b. or g.l.b. of a set may or may not belong to that set.
iii. l.u.b. (supremum) of set may or may not exist like Sup. (N) does not exist. (here, N-set of Natural No.)
iv. g.l.b. (infimum) of a set may or may not exist like Inf (Z) does not exist (here Z-set of integers).

2.1.1 Geometrical Interpretation of Rolle's Theorem

Let the curve $y = f(x)$

i. continuous on $[a, b]$
ii. derivable on (a, b)
iii. $f(a) = f(b)$

This imply that there exists at least one point $c \in (a, b)$ at which tangent is parallel to x-axis.

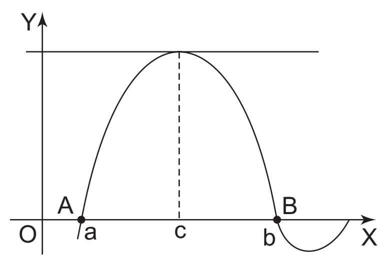

Fig. 2.1

For example:

1. $f(x) = [x]$, greatest integer functions on $[0, 3]$.
f is not continuous at $x = 1, 2, 3$ (break in graph).
$\therefore$ Rolle's theorem does not hold good.

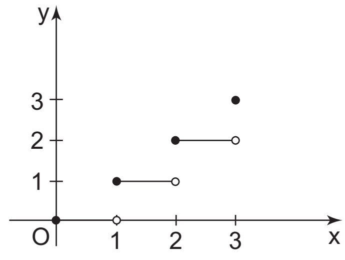

Fig. 2.2

2. $f(x) = x$ in $[-1, 1]$
f is continuous on $[-1, 1]$, f is derivable on $(-1, 1)$ but $f(-1) \neq f(1)$
$\therefore$ Rolle's theorem does not hold good.

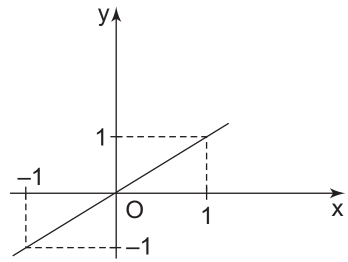

Fig. 2.3

3. $f(x) = |x|$ in $[-1, 1]$
f is continuous on $[-1, 1]$
but f is not derivable at $x = 0$
$\therefore$ Rolle's theorem does not hold good.

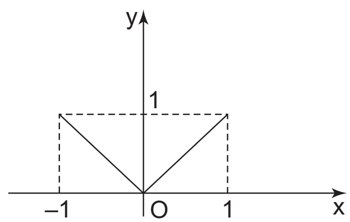

Fig. 2.4

SOME SOLVED EXAMPLES

Example 2.1. *Verify Rolle's theorem for $f(x) = x^3 - 9x^2 + 26x - 24$ in $[2, 4]$.*

Solution. Here, $f(x) = x^3 - 9x^2 + 26x - 24$.

Given $f(x)$ is a polynomial of x and therefore continuous and derivable for all x.

$\Rightarrow$ a. $f(x)$ is continuous on $[2, 4]$

b. $f(x)$ is derivable on $(2, 4)$

c. $f(2) = (2)^3 - 9(2)^2 + 26(2) - 24 = 0$

$f(4) = (4)^3 - 9(4)^2 + 26(4) - 24 = 0$

All three conditions of Rolle's theorem are satisfied.

Hence there must exists atleast one $c \in (2, 4)$ such that $f'(c) = 0$

We have, $f'(x) = 3x^2 - 18x + 26$

$\Rightarrow$ $f'(c) = 3c^2 - 18c + 26$

Thus, $f'(c) = 0$

$$\Rightarrow \qquad c = \frac{18 \pm \sqrt{324 - 312}}{6}$$

$$= 3 \pm \frac{1}{\sqrt{3}}$$

$$c = 3 \pm \frac{1}{\sqrt{3}} \in (2, 4)$$

Hence, Rolle's theorem is verified.

Example 2.2. *Verify Rolle's theorem for f(x) = cos 2x in* $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

Solution. Here $f(x) = \cos 2x$

a. As we know that cosine function is continuous for all values of x and hence $f(x)$ is continuous in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

b. $f'(x) = -2 \sin 2x$ = finite, defined

$\therefore$ $f(x)$ is derivable on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

c. Now,

$$f'\left(\frac{\pi}{4}\right) = \cos 2\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{2} = 0$$

and

$$f'\left(-\frac{\pi}{4}\right) = \cos 2\left(-\frac{\pi}{4}\right) = \cos\left(-\frac{\pi}{2}\right)$$

$$= \cos\frac{\pi}{2} = 0$$

$$\therefore \qquad f\left(-\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right)$$

All the conditions of Rolle's theorem are satisfied.

Hence, there must exists atleast one value of $c \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ such that $f'(c) = 0$

Now $\qquad f'(c) = -2 \sin 2c = 0 \quad \Rightarrow \quad \sin 2c = 0$

$$\sin 2c = \sin 0$$

$$\Rightarrow \qquad 2c = 0 \quad \Rightarrow \quad c = 0$$

$$c = 0 \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$$

Hence, Rolle's theorem is verified.

Example 2.3. *Verify Rolle's theorem for* $f(x) = x(x + 3)e^{-x/2}$ *in [–3, 0].*

Solution. Here $\qquad f(x) = x(x + 3)e^{-x/2}$

a. We know that a function $x(x + 3)$ being a polynomial function and $e^{-x/2}$, the exponential function, both are continuous everywhere.

Thus there product is also continuous in [–3, 0].

b. $$f'(x) = xe^{-x/2} + (x+3)e^{-x/2} - \frac{x(x+3)}{2}\, e^{-x/2}$$
$$= \left(2x+3-\frac{x(x+3)}{2}\right)e^{-x/2}$$
$$= \left(\frac{x+6-x^2}{2}\right)e^{-x/2}$$

$f'(x)$ exists uniquely on $(-3, 0)$ and $f(x)$ is derivable on $(-3, 0)$

c. $$f(-3) = -3\,(-3+3)e^{3/2} = 0$$

and $$f(0) = 0\,(0+3)e^{0/2} = 0$$

Thus $$f(-3) = f(0)$$

Thus, all the conditions of Rolle's theorem are satisfied.

Hence, there must exist atleast one point c in $(-3, 0)$ such that $f'(c) = 0$,

$$f'(c) = \left(\frac{c+6-c^2}{2}\right)e^{-c/2} = 0$$

$\Rightarrow$ $$\frac{c+6-c^2}{2} = 0 \qquad [\because\ e^{-c/2} \neq 0$$

$\Rightarrow$ $$c^2 - c - 6 = 0$$

$\Rightarrow$ $$c = 3, -2$$

Now $$c = -2 \in (-3, 0)$$

Hence Rolle's theorem is verified.

Example 2.4. *Examine the applicability of Rolle's theorem for the function* $f(x) = \begin{cases} -4x+5 & 0 \le x \le 1 \\ 2x-3 & 1 < x \le 2 \end{cases}$.

Solution. Here, $$f(x) = \begin{cases} -4x+5 & 0 \le x \le 1 \\ 2x-3 & 1 < x \le 2 \end{cases}$$

We have, $$f(1) = 1$$

Continuity at, $$x = 1$$

$$R\,f(1) = \lim_{x \to 1^+} (2x-3) = \lim_{h \to 0} 2(1+h) - 3$$
$$= 2 - 3 = -1$$
$$L\,f(1) = \lim_{x \to 1^-} (-4x+5) = \lim_{h \to 0} -4\,(1-h) + 5$$
$$= -4 + 5 = 1$$

Thus, $$R\,f(1) \neq Lf(1)$$

$\therefore$ $f(x)$ is not continuous at $x = 1 \in [0, 2]$

Hence, Rolle's theorem is not applicable.

Example 2.5. *Verify Rolle's theorem for* $f(x) = \sqrt{4-x^2}$ *in* $[-2, 2]$.

Solution. Here $f(x) = \sqrt{4-x^2}$ in $[-2, 2]$. $f(x)$ is a square root of a polynomial of x and therefore continuous for all x.

i. $f(x)$ is continuous on $[-2, 2]$

ii. $f'(x) = \dfrac{-x}{\sqrt{4-x^2}}$ defined everywhere except where $4 - x^2 = 0$ *i.e.*, $x = \pm 2$.

Thus, $f'(x)$ is derivable in $R - \{-2, 2\}$

$\therefore$ $f'(x)$ is derivable on $(-2, 2)$.

iii. Now, $$f(-2) = \sqrt{4-(-2)^2} = \sqrt{4-4} = 0$$

$$f(2) = \sqrt{4-(2)^2} = \sqrt{4-4} = 0$$

Thus, $$f(-2) = f(2)$$

All the conditions of Rolle's theorem are satisfied.

Hence, there must exist atleast one value of $c \in (-2, 2)$ such that $f'(c) = 0$

$$\Rightarrow \qquad f'(c) = \frac{-c}{\sqrt{4-c^2}} = 0$$

$$\Rightarrow \qquad c = 0 \in (-2, 2)$$

Hence, Rolle's theorem is verified.

Example 2.6. *Find a point $c \in (-1, 1)$ using Rolle's theorem for the function*

$$f(x) = \log (x^2 + 2) - \log 3 \text{ in } [-1, 1].$$

Solution. Here $f(x) = \log (x^2 + 2) - \log 3$.

a. As we know that logarithmic functions are continuous for all x and log 3 is a constant, so $f(x)$ is continuous for all x, therefore, continuous in $[-1, 1]$.

b. $$f'(x) = \frac{2x}{x^2+2} \quad (\text{exists for all } x \in \mathbb{R})$$

$\therefore$ $f'(x)$ is derivable on $(-1, 1)$

c. Now, $$f(-1) = \log [(-1)^2 + 2] - \log 3$$
$$= \log 3 - \log 3 = 0$$

and $$f(1) = \log [(1)^2 + 2] - \log 3$$
$$= \log 3 - \log 3 = 0$$

$$\Rightarrow \qquad f(-1) = f(1)$$

Thus, all conditions of Rolle's theorem are satisfied.

Hence there must exist atleast one $c \in (-1, 1)$ such that $f'(c) = 0$

$$f'(c) = \frac{2c}{c^2+2} = 0$$

$$\Rightarrow \qquad c = 0 \in (-1, 1).$$

EXERCISE 2.1

1. Verify Rolle's theorem for the following functions in the given intervals:

a. $f(x) = x^3 - 6x^2 + 11x - 6$ in $[1, 3]$

b. $f(x) = x^3 + 3x^2 - 24x - 80$ in $[-4, 5]$

c. $f(x) = \dfrac{x^2-3x-4}{x-5}$ in $[-1, 4]$

d. $f(x) = \cos 2\left(x - \dfrac{\pi}{4}\right)$ in $\left[0, -\dfrac{\pi}{2}\right]$

e. $f(x) = \sin x - \sin 2x$ in $[0, \pi]$

f. $f(x) = e^{1-x^2}$ in $[-1, 1]$

g. $f(x) = \dfrac{\sin x}{e^x}$ in $[0, \pi]$

h. $f(x) = e^x \cos x$ in $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

i. $f(x) = \tan x$ in $[0, \pi]$

j. $f(x) = (x^2 - 4x + 3)e^{2x}$ in $[1, 3]$

2. Examine the applicability of Rolle's theorem for the following functions:

a. $f(x) = (x-1)^{2/5}$ in $[0, 3]$

b. $f(x) = \begin{cases} x^2+1, & 0 \le x \le 1 \\ 3-x, & 1 < x \le 2 \end{cases}$

c. $f(x) = |x|$ in $[-1, 1]$

Answers

1. a. $c = 2 \pm \dfrac{1}{\sqrt{3}}$ b. $c = 2$ c. $c = 5 - \sqrt{6}$ d. $c = \dfrac{\pi}{4}$

e. $c = \cos^{-1}\dfrac{1 \pm \sqrt{33}}{8}$ f. $c = 0$ g. $c = \dfrac{\pi}{4}$ h. $c = \dfrac{\pi}{4}$

i. Not applicable j. $c = \dfrac{3+\sqrt{5}}{2}$

2. a. Not applicable b. Not applicable c. Not applicable

2.1.2 Lagrange's Mean Value Theorem

Statement: If a function $f: [a, b] \to \mathbb{R}$ be such that

i. $f(x)$ is continuous on closed interval $[a, b]$

ii. $f(x)$ is derivable on open interval (a, b),

then there exist atleast one point $c \in (a, b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Proof: Let us define a function $\phi : [a, b] \to \mathbb{R}$ by $\phi(x) = f(x) + Ax, x \in (a, b)$

where A is a constant to be determined such that

$$\phi(a) = \phi(b) \qquad ...(1)$$

Now,

$$\phi(a) = f(a) + Aa$$

$$\phi(b) = f(b) + Ab$$

Using (1), we have

$$f(a) + Aa = f(b) + Ab$$

$$A(a-b) = f(b) - f(a)$$

$$A = \frac{f(b)-f(a)}{a-b} \qquad ...(2)$$

Now,

i. ϕ is continuous on $[a, b]$, since f is continuous on $[a, b]$ and Ax is polynomial in x is continuous on $[a, b]$.

ii. ϕ is derivable on (a, b), since f is derivable at each point of (a, b) and also Ax.

iii. $$\phi(a) = \phi(b)$$

$\phi(x)$ satisfies all three conditions of Rolle's theorem.

Hence there exists atleast one $c \in (a, b)$ such that $\phi'(c) = 0$

$$\phi(x) = f(x) + Ax$$

$$\Rightarrow \quad \phi'(x) = f'(x) + A$$

$$\Rightarrow \quad \phi'(c) = f'(c) + A$$

Now, $$\phi'(c) = 0$$

$$\Rightarrow \quad f'(c) + A = 0 \quad \Rightarrow \quad f'(c) = -A$$

Using (2), we have, $$f'(c) = \frac{f(b) - f(a)}{b - a}, c \in (a, b)$$

This completes the proof of the theorem.

2.1.3 Geometrical Interpretation of Lagrange's Mean Value Theorem

Let a function *f* has a graph which is

i. continuous on $[a, b]$

ii. differentiable on (a, b)

As curve *AB* has a tangent at every point, then there exist a point on the curve other than *A* and *B* where tangent is parallel to line segment joining the points $(a, f(a))$ and $(b, f(b))$.

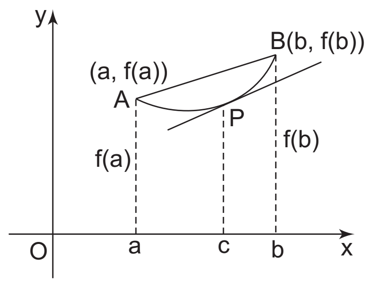

Fig. 2.5

Remarks: Rolle's theorem is a special case of Lagrange's mean value theorem.

In addition with two conditions of mean value theorem, if

$$f(a) = f(b)$$

then $$f(b) - f(a) = 0$$

and hence $$f'(c) = 0$$

In Geometrical Interpretation, there is a point on curve at which tangent is parallel to x-axis.

SOME SOLVED EXAMPLES

Example 2.7. *Verify Lagrange's mean value theorem for f(x) =* $x + \frac{1}{x}$ *in [1, 3].*

Solution. Here, $f(x) = x + \frac{1}{x}$

i. $f(x)$ is polynomial in x and continuous for all values of $x \in \mathbb{R} - \{0\}$

$\therefore$ $f(x)$ is continuous on $[1, 3]$

ii. $f'(x) = 1 - \frac{1}{x^2}$ exists for all $x \in (1, 3)$

$\therefore$ $f(x)$ is derivable in $(1, 3)$

Both conditions of Lagrange's mean value theorem are satisfied.

Hence, there must exist atleast one $c \in (1, 3)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c) \qquad \text{...(1) [Here, } b = 3, a = 1]$$

Putting $f(b), f(a), f'(c)$ in (1), we have

$$\frac{\frac{10}{3}-2}{3-1} = 1 - \frac{1}{c^2}$$

$$\Rightarrow \qquad \frac{1}{c^2} = \frac{1}{3} \quad \Rightarrow \quad c = \pm\sqrt{3}$$

$$c = \sqrt{3} \in (1, 3)$$

Hence, Lagrange's mean value theorem is verified.

Example 2.8. *Show that* $x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}, x > 0.$

Solution. Consider $\quad f(x) = \log(1+x) - \left(x - \frac{x^2}{2}\right)$

Differentiating w.r.t. x,

$$f'(x) = \frac{1}{1+x} - (1-x)$$

$$= \frac{x^2}{1+x} > 0 \qquad [\because \quad x > 0]$$

Hence $f(x)$ is increasing function for all $x > 0$

Also $f(0) = 0$

Hence, $f(x) > 0$ for $x > 0$

Thus, $$\log(1+x) > x - \frac{x^2}{2} \qquad \text{...(1)}$$

Let $$g(x) = x - \frac{x^2}{2(1+x)} - \log(1+x)$$

Differentiating w.r.t x,

$$g'(x) = 1 - \frac{2x+x^2}{2(1+x)^2} - \frac{1}{1+x}$$

$$= \frac{x^2}{2(1+x)^2} = \frac{1}{2}\left(\frac{x}{1+x}\right)^2 > 0$$

$\therefore \quad g(x)$ is an increasing function for all $x > 0$

Also $g(0) = 0$

Hence, $g(x) > 0$ for $x > 0$

Thus, $$x - \frac{x^2}{2(1+x)} > \log(1+x) \qquad \text{...(2)}$$

From (1) and (2), we have

$$x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}, x > 0.$$

Example 2.9. *Examine the validity of Lagrange's mean value theorem for the function*

$$f(x) = \begin{cases} 1+3x, & x \le 1 \\ 2x^2+2, & x > 1 \end{cases} \text{ in } [0, 3].$$

Solution. Here $f(x) \begin{cases} 1+3x, & x \le 1 \\ 2x^2+2, & x > 1 \end{cases}$ in [0, 3]

i. $f(x)$ is continuous on $[0, 3] - \{1\}$ being a polynomial function.

Continuity at $x = 1$

$$R f(1) = \lim_{x \to 1^+} 2x^2 + 2 = \lim_{h \to 0} 2(1+h)^2 + 2 = 4$$

$$L f(1) = \lim_{x \to 1^-} 1 + 3x = \lim_{h \to 0} 1 + 3(1-h) = 4$$

Also, $f(1) = 4$

We have $R f(1) = L f(1) = f(1)$

$\Rightarrow$ $f(x)$ is continuous at $x = 1$

$\therefore$ $f(x)$ is continuous on [0, 3]

ii. $$f'(x) = \begin{cases} 3, & x \le 1 \\ 4x, & x > 1 \end{cases} \text{ in } [0, 3]$$

Differentiability at $x = 1$

$$R f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{4(1+h) - 4}{h} = 4$$

$$L f'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{3-4}{h} = \text{does not exist}$$

$f'(x)$ does not exist for $x = 1 \in (0, 3)$

$\Rightarrow$ $f(x)$ is not derivable on (0, 3)

Hence Lagrange's mean value theorem is not applicable.

Example 2.10. *Verify Lagrange's mean value theorem for $f(x) = \cos x$ in $\left[0, \frac{\pi}{2}\right]$.*

Solution. Here $f(x) = \cos x$

i. As we know, cosine function is continuous for all value of x.

$\therefore$ $f(x)$ is continuous on $\left[0, \frac{\pi}{2}\right]$

ii. $f'(x) = -\sin x$ (finite and definite)

$\therefore$ $f(x)$ is derivable in $(0, \pi/2)$

Both conditions of Lagrange's mean value theorem are satisfied.

Hence there must exist atleast one $c \in \left(0, \frac{\pi}{2}\right)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c), \text{ here } b = \pi/2, a = 0$$

$$f(\pi/2) = \cos \pi/2 = 0, f(0) = \cos 0 = 1$$

Putting values,

$$\Rightarrow \quad \frac{0-1}{\frac{\pi}{2}-0} = -\sin c$$

$$\Rightarrow \quad c = \sin^{-1}(2/\pi)$$

$$\Rightarrow \quad c = \sin^{-1}(0.636) \in \left(0, \frac{\pi}{2}\right)$$

Hence Lagrange's mean value theorem is satisfied.

Example 2.11. *Show that* $\frac{x}{1+x^2} < \tan^{-1} x < x, x > 0.$

Solution. Consider $\quad f(x) = \tan^{-1}x - \frac{x}{1+x^2}$

Differentiating w.r.t. x, we get

$$f'(x) = \frac{1}{1+x^2} - \frac{1-x^2}{(1+x^2)^2}$$

$$= \frac{2x^2}{(1+x^2)^2} > 0 \; \forall \; x > 0$$

Hence $f(x)$ is a increasing function for all $x > 0$

Also $\quad f(0) = 0 \qquad$ (as $\tan^{-1} 0 - 0 = 0$)

Hence $\quad f(x) > 0 \; \forall \; x > 0$

Thus, $\quad \tan^{-1}x > \frac{x}{1+x^2} \qquad$ ...(1)

Let $\quad g(x) = x - \tan^{-1}x$

Differentiating w.r.t. x,

$$g'(x) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2} > 0 \; \forall \; x > 0$$

$\therefore$ $g(x)$ is an increasing function for all $x > 0$

Also, $\quad g(0) = 0 - \tan^{-1} 0 = 0$

Hence $\quad g(x) > 0 \; \forall \; x > 0$

Thus, $\quad x > \tan^{-1} x \; \forall \; x > 0 \qquad$ ...(2)

Thus, from (1) and (2), we have

$$\frac{x}{1+x^2} < \tan^{-1} x < x, x > 0.$$

EXERCISE 2.2

1. Verify Lagrange's Mean Value theorem for the following functions in given intervals.

a. $f(x) = 2x^2 - 3x + 1$ in $[1, 3]$

b. $f(x) = x(x-1)(x-2)$ in $\left[0, \frac{1}{2}\right]$

c. $f(x) = \dfrac{1}{4x-1}$ in $[1, 4]$

d. $f(x) = \sqrt{25 - x^2}$ in $[-3, 4]$

e. $f(x) = \log x$ in $[1, e]$

f. $f(x) = x - 2\sin x$ in $[-\pi, \pi]$

2. Examine the applicability of Lagrange's mean value theorem for following functions:

a. $f(x) = |x|$ in $[-1, 1]$

b. $f(x) = \beta$ (constant function) in $[a, b]$

c. $f(x) = x^{1/3}$ in $[-1, 1]$

d. $f(x) = |x + 2|$ in $[-3, 4]$

3. Using Lagrange's Mean Value theorem, prove that

a. $\dfrac{x^2}{2} < x - \log(1+x) < \dfrac{x^2}{2(1+x)}$ on $[-1, 0]$

b. $\dfrac{x}{1+x} < \log(1+x) < x, x > 0$

c. $1 + x < e^x < 1 + xe^x$ for all $x > 0$

4. Show that $\dfrac{y-x}{1+y^2} < \tan^{-1} y - \tan^{-1} x < \dfrac{y-x}{1+x^2}$ if $0 < x < y$ and deduce that

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$$

Answers

1. a. $c = 2$ b. $c = \dfrac{6-\sqrt{21}}{6}$ c. $c = \dfrac{1+3\sqrt{5}}{4}$ d. $c = \pm\dfrac{1}{\sqrt{2}}$

e. $c = e - 1$ f. $c = \pm\pi/3$

2. a. Not applicable b. Applicable c. Not applicable d. Not applicable

2.1.4 Cauchy Mean Value Theorem

Statement: Let the function $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be such that

i. f and g are both continuous on $[a, b]$

ii. f and g are both differentiable on (a, b)

iii. $g'(x) \neq 0$ for all $x \in (a, b)$, then there exist atleast one point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Proof: Suppose $g(a) = g(b)$, then g would satisfy all the conditions of Rolle's theorem.

So, there exist atleast one point $c \in (a, b)$ such that $g'(c) = 0$

But this contradicts the given fact that $g'(x) \neq 0 \ \forall\ x \in (a, b)$, so our supposition is wrong and $g(a) \neq g(b)$.

Now define a function $\phi : [a, b] \to \mathbb{R}$ by $\phi(x) = f(x) + Ag(x)$, $x \in (a, b)$ where A is constant and to be determined in such a way that,

$$\phi(a) = \phi(b) \qquad ...(1)$$

Now, $\phi(a) = f(a) + Ag(a)$

$$\phi(b) = f(b) + Ag(b)$$

Using (1), we have

$$f(a) + Ag(a) = f(b) + Ag(b)$$

$$\Rightarrow \quad A[g(a) - g(b)] = f(b) - f(a)$$

$$\Rightarrow \quad A = \frac{f(b) - f(a)}{g(a) - g(b)} \qquad ...(2)$$

Now,

i. ϕ is continuous on $[a, b]$, since f and g both are continuous on $[a, b]$ and A being a constant is also continuous on $[a, b]$

ii. ϕ is derivable on (a, b), since f and g both are differentiable on (a, b) and A being a constant is also derivable on (a, b)

iii. Also, $\phi(a) = \phi(b)$

Thus, $\phi(x)$ satisfies all three conditions of Rolle's theorem.

Hence there exist atleast one $c \in (a, b)$ such that $\phi'(c) = 0$

$$\phi(x) = f(x) + Ag(x)$$

$$\Rightarrow \quad \phi'(x) = f'(x) + Ag'(x)$$

$$\Rightarrow \quad \phi'(c) = f'(c) + Ag'(c)$$

Now, $\phi'(c) = 0$

$$\Rightarrow \quad f'(c) + Ag'(c) = 0$$

$$\Rightarrow \quad f'(c) = -Ag'(c)$$

$$\Rightarrow \quad \frac{f'(c)}{g'(c)} = -A$$

Using (2), we have, $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}, c \in (a, b)$

Hence, theorem is proved.

> **Remark:** Lagrange's mean value theorem is a particular case of Cauchy's mean value theorem by taking $g(x) = x$, $x \in [a, b]$.

2.1.5 Geometrical Interpretation of Cauchy's Mean Value Theorem

Let $x = f(t)$ and $y = g(t)$ be parametric curve, $t \in (a, b)$

i. f, g continuous on $[a, b]$

ii. f, g derivable on (a, b)

iii. $g'(x) \neq 0$ on (a, b)

then there exist atleast one $c \in (a, b)$ at which tangent is parallel to AB.

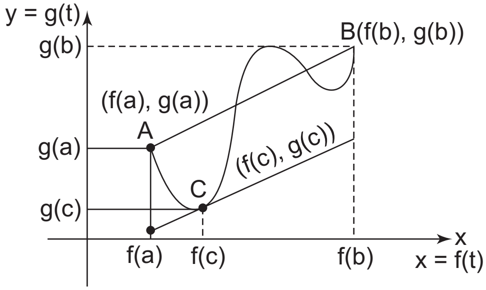

Fig. 2.6

SOME SOLVED EXAMPLES

Example 2.12. *Verify Cauchy's mean value theorem for* $f(x) = e^x$ *and* $g(x) = e^{-x}$ *on [0, 1].*

Solution. Here, $f(x) = e^x, g(x) = e^{-x}$

i. f and g are continuous function on [0, 1]

ii. $f'(x) = e^x, g'(x) = -e^{-x}$ are differentiable on (0, 1)

iii. $g'(x) = -e^{-x} \neq 0 \ \forall \ x \in (0, 1)$,

then there exist atleast one $c \in (0, 1)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \qquad [\text{here } b = 1, a = 0] \qquad ...(1)$$

So $f(1) = e^1 = e, f(0) = e^0 = 1$

and $g(1) = e^{-1} = \dfrac{1}{e}, g(0) = e^{-0} = 1$

Putting all values in (1), we have

$$\frac{e-1}{\frac{1}{e}-1} = \frac{e^c}{-e^{-c}}$$

$\Rightarrow \quad 1 = e^{2c-1}$

$\Rightarrow \quad e^0 = e^{2c-1}$

$\Rightarrow \quad 0 = 2c - 1 \quad \Rightarrow \quad c = \dfrac{1}{2} \in (0, 1)$

Hence Cauchy's mean value theorem is verified.

Example 2.13. *Let the function f be continuous in [a, b] and derivable in (a, b). Show that there exists a number c in (a, b) such that* $2c\,[f(a) - f(b)] = f'(c)\,[a^2 - b^2]$.

Solution. i. f is continuous in $[a, b]$

ii. f' is derivable in (a, b)

Both conditions of Lagrange's mean value theorem are satisfied.

Hence there exist atleast one $c \in (a, b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$\Rightarrow \qquad \dfrac{f(a)-f(b)}{a-b} = f'(c)$

$\Rightarrow \qquad f(a) - f(b) = f'(c)\,(a - b)$ ...(1)

Now, given $2c\,[f(a) - f(b)] = f'(c)\,[a^2 - b^2]$

Using (1), we have

$$2c\,[(a - b)\,f'(c)] = f'(c)\,[(a - b)\,(a + b)]$$

$\Rightarrow \qquad 2c = a + b$

$\Rightarrow \qquad c = \dfrac{a+b}{2} \in (a, b)$

Hence, there exist a number $c \in (a, b)$.

Example 2.14. *Find 'c' in the Cauchy mean value theorem for the function*

$$f(x) = \frac{1}{x},\ g(x) = x^2 - 4 \text{ in } [1, 2] \text{ using } \sqrt[3]{3} = 1.44.$$

Solution. Here, $f(x) = \dfrac{1}{x}$, $g(x) = x^2 - 4$ in $[1, 2]$

i. f is continuous function for all $x \in R - \{0\}$ [$\because$ f is not defined at $x = 0$]

and g being a polynomial function is continuous everywhere.

$\therefore$ $f(x)$ and $g(x)$ are continuous function on $[1, 2]$

ii. $f'(x) = -\dfrac{1}{x^2}$, $g'(x) = 2x$ are derivable on $(1, 2)$

iii. $g'(x) = 2x \neq 0\ \forall\ x \in (1, 2)$,

then there exist atleast one $c \in (1, 2)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \qquad ...(1) \quad \text{here } a = 1, b = 2$$

So $\qquad f(2) = 1/2, f(1) = 1, g(2) = 0, g(1) = -3$

Putting all values in (1), we have

$$\frac{\frac{1}{2}-1}{0-(-3)} = \frac{-1/c^2}{2c}$$

$\Rightarrow \qquad c^3 = 3$

$\Rightarrow \qquad c = \sqrt[3]{3} = 1.44 \in (1, 2).$

Example 2.15. *Verify Cauchy mean value theorem for the function $f(x) = x^2$, $g(x) = x^4$ in $[a, b]$, where $a > 0, b > 0$.*

Solution. Here $\qquad f(x) = x^2, g(x) = x^4$

i. since f and g are polynomial functions of x, therefore continuous everywhere.

$\therefore$ $f(x)$ and $g(x)$ are continuous on $[a, b]$

ii. $\qquad f'(x) = 2x, g'(x) = 4x^3$

$f'(x)$ and $g'(x)$ are again polynomial function and hence derivable everywhere.

$\therefore$ $f'(x)$ and $g'(x)$ are derivable on (a, b)

iii. $\qquad g'(x) = 4x^3 \neq 0\ \forall\ x \in (a, b), [a > 0, b > 0]$

Thus f and g satisfies all the conditions of Cauchy mean value theorem.

$\therefore$ there must exist atleast one $c \in (a, b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \qquad \text{[here } a = a, b = b\text{]}$$

So, $f(b) = b^2, f(a) = a^2, g(a) = a^4, g(b) = b^4$

$$\Rightarrow \quad \frac{b^2 - a^2}{b^4 - a^4} = \frac{2c}{4c^3}$$

$$\Rightarrow \quad \frac{1}{b^2 + a^2} = \frac{1}{2c^2}$$

$$\Rightarrow \quad c^2 = \frac{b^2 + a^2}{2}$$

$$\Rightarrow \quad c = \pm\sqrt{\frac{a^2 + b^2}{2}} \in (a, b)$$

Hence, Cauchy mean value theorem is verified.

EXERCISE 2.3

1. Verify Cauchy's Mean Value Theorem for the following functions:

a. $f(x) = \sin x, g(x) = \cos x$ in $\left[-\frac{\pi}{2}, 0\right]$

b. $f(x) = x^2, g(x) = x^3$ in $[1, 2]$

c. $f(x) = \sqrt{x}, g(x) = \frac{1}{\sqrt{x}}$ in $[1, 3]$

d. $f(x) = \log x, g(x) = \frac{1}{x}$ in $[1, e]$

e. $f(x) = (1 + x)^{3/2}, g(x) = \sqrt{1+x}$ in $\left[0, \frac{1}{2}\right]$

2. If f' and g' are continuous and differentiable on $[a, b]$, then show that $a < c < b$

$$\frac{f(b)-f(a)-(b-a)f'(a)}{g(b)-g(a)-(b-a)g'(a)} = \frac{f''(c)}{g''(c)}.$$

3. Show that $\dfrac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta, 0 < \alpha < \theta < \beta < \dfrac{\pi}{2}$.

Answers

1. a. $c = -\frac{\pi}{4}$ b. $c = 14/9$ c. $c = \sqrt{3}$ d. $c = \frac{e}{e-1}$

e. $c = \frac{\sqrt{6}-1}{\sqrt{6}}$

INTERESTING FACTS

- Rolle's Theorem establishes a connection between continuity and differentiability.
- Mean value theorem is even used to check the accuracy of a speedometer.
- It specifies the existence of a point where the derivative vanishes.

VIDEO REFERENCES

Rolle's Theorem

Rolle's Theorem & Lagrange Mean Value Theorem (MVT)

Mean Value Theorems

USES OF ICT

- https://www.mathwarehouse.com/calculus/derivatives/what-is-rolles-theorem.php

APPLICATIONS TO REAL LIFE

- If the average speed during a journey from A to B was say 50 kms/hour, then there had to be a time when the instantaneous speed was 50 kms/hour as well (that is the maximum)
- The rate of change in timings of the sunset, over the seasons.
- When a ball is thrown upwards in the air, its velocity becomes zero at some point of time. Rolle's Theorem explains that the velocity of ball becomes zero at some point of time.
- LMVT is used to issue "*challan*" for speeding.

2.2 TAYLOR'S THEOREM

Taylor's theorem is an extension of mean value theorem as mean value theorem relates the value of function and its first order derivative but Taylor's theorem relates the value of function and its 'higher order derivatives'.

2.2.1 Taylor's Theorem with Lagrange's form of Remainder

Statement: If a function $f : [a, a + h] \to \mathbb{R}$ be such that

i. $f, f', f'', ..., f^{n-1}$ are continuous function of x in the closed interval $[a, a + h]$.

ii. $f^n(x)$ exists in the open interval $(a, a + h)$ then there exists atleast one real number θ; $0 < \theta < 1$, such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(\theta h + a)$$

Proof: Consider a function $\phi : [a, a + h] \to \mathbb{R}$ in such a way that

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + ... \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x) + \frac{(a+h-x)^n}{n!} A \qquad ...(1)$$

where A is a constant to be chosen such that

$$\phi(a) = \phi(a+h) \qquad ...(2)$$

Now putting $x = a$ and $x = a + h$ in (1), we have

$$\phi(a) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} A$$

and $\phi(a+h) = f(a+h)$

Putting these values in (2), we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} A \qquad ...(3)$$

Now,

i. $\phi(x)$ is continuous on $[a, a+h]$, since $f(x), f'(x), f''(x), ..., f^{n-1}(x)$ are continuous on $[a, a+h]$ and $(a+h-x), (a+h-x)^2, ..., (a+h-x)^n$ being polynomials are also continuous on closed interval $[a, a+h]$. Also the algebraic sum of continuous functions is continuous.

ii. $\phi(x)$ is derivable in $(a, a+h)$, since $f(x), f'(x), ... + f^{n-1}(x)$ are all derivable on $(a, a+h)$. Also $(a+h-x), (a+h-x)^2, ... (a+h-x)^n$ being polynomials are derivable in the open interval $(a, a+h)$.

iii Also, $\phi(a) = \phi(a+h)$,

$\therefore$ $\phi(x)$ satisfies all the three conditions of Rolle's theorem in $[a, a+h]$. Hence there exists atleast one real number θ, $0 < \theta < 1$, such that

$$\phi'(a+\theta h) = 0 \qquad ...(4)$$

Differentiating (1) w.r.t. x, we get

$$\phi'(x) = f'(x) + (a+h-x) f''(x) - f'(x) + \frac{(a+h-x)^2}{2!} f'''(x) + \frac{2(a+h-x)}{2!} (-1) f''(x)$$

$$+ ... + ... + \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - \frac{(n-1)(a+h-x)^{n-2}}{(n-1)!} f^{n-1}(x)$$

$$+ \frac{n(a+h-x)^{n-1}(-1)}{n!} A$$

or

$$\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - \frac{(a+h-x)^{n-1}}{(n-1)!} A$$

$$= \frac{(a+h-x)^{n-1}}{(n-1)!} [f^n(x) - A]$$

Putting $x = a + \theta h$

$$\phi'(a+\theta h) = \frac{[h(1-\theta)]^{n-1}}{(n-1)!} [f^n(a+\theta h) - A]$$

But $\phi'(a+\theta h) = 0$ [From 4]

$\Rightarrow$ $f^n(a+\theta h) - A = 0 \Rightarrow A = f^n(a+\theta h)$ [$\because$ $1-\theta \neq 0$ and $h \neq 0$]

From (3),

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h)$$

which is the required result of the theorem.

Here, the $(n + 1)^{th}$ term *i.e.*, $\frac{h^n}{n!} f^n(a + \theta h)$ is called the Lagrange's form of remainder after nth term.

2.2.2 Maclaurin's Theorem with Lagrange's Form of Remainder

Statement: If a function $f(x)$ defined in closed interval $[0, x]$ is such that

i. $f(x), f'(x), f''(x), ..., f^{n-1}(x)$ are continuous in closed interval $[0, x]$

ii. $f^n(x)$ exists in open interval $(0, x)$, then there exist atleast one real number $\theta, 0 < \theta < 1$ such that

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + ... + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x)$$

Proof: From Taylor's theorem, we have

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a + \theta h)$$

In this expression, put $a = 0, h = x$, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + ... + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x), 0 < \theta < 1$$

The above expression is the required Maclaurin's theorem with Lagrange's form of remainder.

2.2.3 Taylor's Theorem with Cauchy's Form of Remainder

Statement: If a function $f: [a, a + h] \to \mathbb{R}$ be such that

i. $f, f', f'', ..., f^{n-1}$ are all continuous function of x in the closed interval $[a, a + h]$

ii. $f^n(x)$ exists in the open interval $(a, a + h)$, then there exists atleast one real number θ; $0 < \theta < 1$, such that

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n (1-\theta)^{n-1}}{(n-1)!} f^n(a + \theta h)$$

Proof: Consider a function $\phi : [a, a + h] \to \mathbb{R}$ by

$$\phi(x) = f(x) + (a + h - x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + ... + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x) + (a + h - x) . A \quad ...(1), x \in (a, a + h)$$

where A is constant to be chosen such that $\phi(a) = \phi(a + h)$

Putting $x = a$ in (1), $\phi(a) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + hA$ ...(2)

and $\phi(a + h) = f(a + h)$

Putting $x = a + h$, in (1), we have

$$\phi(a + h) = f(a + h) + 0 + 0 + ... + 0 = f(a + h) \quad ...(3)$$

Now, $\phi(a + h) = \phi(a)$

Then, from (2) and (3), we have

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + hA \quad ...(4)$$

Now,

i. ϕ is continuous on $[a, a + h]$, since $f, f', f'', \ldots f^{n-1}$ are continuous on $[a, a + h]$ and also $(a + h - x), (a + h - x)^2, \ldots, (a + h - x)^{n-1}$ being polynomials are continuous on $[a, a + h]$

ii. ϕ is differentiable in $[a, a + h]$, since $f, f', f'', \ldots f^{n-1}$ are all differentiable in $(a, a + h)$ and also $(a + h - x), (a + h - x)^2, \ldots, (a + h - x)^{n-1}$ being polynomial are derivable in $(a, a + h)$

iii. Also $\phi(a) = \phi(a + h)$

Now, $\phi(x)$ satisfies all three conditions of Rolle's theorem.

Hence there exist atleast one real number θ, $0 < \theta < 1$, such that $\phi'(a + \theta h) = 0$

Differentiating both sides of (1) w.r.t. x, we have,

$$\phi'(x) = f'(x) + [(a + h - x) f''(x) - f'(x)] + \frac{1}{2!} [2(a + h - x) f''(x) (-1)$$
$$\frac{(a+h-x)^2}{2!} f'''(x)] + \ldots + \ldots + \frac{1}{(n-1)!} [(a + h - x)^{n-1} f^n(x)$$
$$- (n - 1) (a + h - x)^{n-2} f^{n-1}(x)] - A$$

or
$$\phi'(x) = \frac{(a+h-x)^{n-1} f^n(x)}{(n-1)!} - A$$

Putting $x = a + \theta h$, we have

$$\phi'(a + \theta h) = \frac{(a+h-a-\theta h)^{n-1} f^n(a+\theta h)}{(n-1)!} - A$$
$$= \frac{[h(1-\theta)]^{n-1} f^n(a+\theta h)}{(n-1)!} - A$$
$$= \frac{h^{n-1}}{(n-1)!} (1 - \theta)^{n-1} f^n(a + \theta h) - A$$

But $\phi'(a + \theta h) = 0$

$$\Rightarrow \frac{h^{n-1}}{(n-1)!} (1 - \theta)^{n-1} f^n(a + \theta h) - A = 0$$

or
$$A = \frac{h^{n-1}}{(n-1)!} (1 - \theta)^{n-1} f^n (a + \theta h)$$

Putting this value of 'A' in (4), we have

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1} (a) + h\left[\frac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1} f^n(a+\theta h)\right]$$

i.e., $$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^n}{(n-1)!} f^{n-1} (a) + \frac{h^n}{(n-1)!} (1 - \theta)^{n-1} f^n(a + \theta h)$$

is the required form of Taylor's theorem with Cauchy's form of remainder.

Here, $\frac{h^n}{(n-1)!} (1 - \theta)^{n-1} f^n(a + \theta h)$ is called the Cauchy form of remainder after n^{th} term.

2.2.4 Maclaurin's Theorem with Cauchy's Form of Remainder

Statement: If a function $f(x)$ defined on $[0, x]$ is such that

i. $f(x), f'(x), f''(x), \ldots, f^{n-1}(x)$ are continuous function in $[0, x]$

ii. $f^{n}(x)$ exists in $(0, x)$,

then there exist atleast one real number θ, $0 < \theta < 1$ such that

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n (1-\theta)^{n-1}}{(n-1)!} f^{n}(\theta x), 0 < \theta < 1$$

Proof: From Taylor's theorem with Cauchy form of remainder, we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{(n-1)!} (1-\theta)^{n-1} f^{n}(a+\theta h)$$

Put $a = 0$ and $h = x$ in above theorem, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{(n-1)!} (1-\theta)^{n-1} f^{n}(\theta x)$$

which is the Maclaurin's theorem with Cauchy's form of remainder.

SOME SOLVED EXAMPLES

Example 2.16. *The expansion of the function $f(x) = (1-x)^{7/2}$ is given by $f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(\theta x)$. Find the value of θ as $x \to 1$.*

Solution. Here,

$$f(x) = (1-x)^{7/2}$$

$$f'(x) = -\frac{7}{2}(1-x)^{5/2}$$

$$f''(x) = \frac{35}{4}(1-x)^{3/2}$$

$$f'''(x) = -\frac{105}{8}(1-x)^{1/2}$$

Finding all above derivatives at $x = 0$, we have

$$f(0) = 1, f'(0) = \frac{-7}{2}, f''(0) = \frac{35}{4}$$

$$f'''(\theta x) = -\frac{105}{8}(1-\theta x)^{1/2}$$

Now,
$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(\theta x)$$

$$\Rightarrow \quad (1-x)^{7/2} = 1 - \frac{7}{2}x + \frac{35}{8}x^2 - \frac{105}{48}(1-\theta x)^{1/2} . x^3$$

When $x \to 1$, we have
$$0 = 1 - \frac{7}{2} + \frac{35}{8} - \frac{105}{48}(1-\theta)^{1/2}$$

$$\Rightarrow \qquad \frac{105}{48}(1-\theta)^{1/2} = \frac{15}{8}$$

$$\Rightarrow \qquad (1-\theta)^{1/2} = \frac{15}{8}\times\frac{48}{105}$$

$$\Rightarrow \qquad (1-\theta)^{1/2} = \frac{6}{7}$$

Squaring on both sides

$$1-\theta = \frac{36}{49}$$

$$\Rightarrow \qquad \theta = 1-\frac{36}{49}$$

$$\Rightarrow \qquad \theta = \frac{13}{49} \quad \textbf{Answer}$$

Example 2.17. *Show that for every value of x,*

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + (-1)^{n+1}\frac{x^{2n+1}}{(2n+1)!}\sin(\theta x).$$

Solution. Given, $f(x) = \cos x$

$$\therefore \qquad f'(x) = -\sin x = \cos\left(x+\frac{\pi}{2}\right)$$

$$f''(x) = -\cos x = \cos(x+\pi)$$

$$f'''(x) = \sin x = \cos\left(\frac{3\pi}{2}+x\right)$$

$$f^{iv}(x) = \cos x = \cos(2\pi + x)$$

....

....

$$f^{n}(x) = \cos\left[x+\frac{n\pi}{2}\right]$$

$$f^{2n-1}(x) = \cos\left[x+(2n-1)\frac{\pi}{2}\right]$$

$$f^{2n}(x) = \cos\left[x+2n\,.\,\frac{\pi}{2}\right]$$

$$= \cos[x+n\pi]$$

$$f^{2n+1}(x) = \cos\left[x+(2n+1)\frac{\pi}{2}\right]$$

$$f^{2n+1}(\theta x) = \cos\left[\theta x+(2n+1)\frac{\pi}{2}\right]$$

So, $f(0) = \cos 0 = 1, \qquad f'(0) = -\sin 0 = 0$

$$f''(0) = -\cos 0 = -1, \qquad f'''(0) = \sin 0 = 0$$

$$f^{iv}(0) = \cos 0 = 1$$

....

....

$$f^{2n-1}(0) = \cos\left[(2n-1)\frac{\pi}{2}\right]$$

$$= \cos\left(n\pi - \frac{\pi}{2}\right) = 0$$

$$f^{2n}(0) = \cos n\pi = \begin{cases} 1, & n = \text{even} \\ -1, & n = \text{odd} \end{cases} = (-1)^n$$

$$f^{2n+1}(\theta x) = \cos\left[\theta x + n\pi + \frac{\pi}{2}\right]$$

$$= \begin{cases} -\sin\theta x, & n = \text{even} \\ \sin\theta x, & n = \text{odd} \end{cases} = (-1)^{n+1}\sin\theta x$$

By Maclaurin's theorem, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ... + \frac{x^{2n}}{(2n)!}f^{2n}(0) + \frac{x^{2n+1}}{(2n+1)!}f^{2n+1}(\theta x)$$

On substituting the values, we have

$$\cos x = 1 + x\,.\,0 + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + ... + \frac{x^{2n}}{(2n)!}(-1)^n + \frac{x^{2n+1}}{(2n+1)!}(-1)^{n+1}\sin\theta x$$

i.e., $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... + (-1)^n\frac{x^{2n}}{(2n)!} + (-1)^{n+1}\frac{x^{2n+1}}{(2n+1)!}\sin\theta x$$ **Proved.**

Example 2.18. *If a function f is such that f′ is continuous on [a, b] and derivable on (a, b). Show that there exist a real number θ, 0 < θ < 1, such that*

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''[a + \theta(b-a)].$$

Solution. Consider the function

$$\phi(x) = f(x) + (b-x)f'(x) + (b-x)^2 A \qquad ...(1)$$

where A is constant to be chosen such that

$$\phi(a) = \phi(b)$$

Now, $$\phi(a) = f(a) + (b-a)f'(a) + (b-a)^2 A \qquad ...(2)$$

and $$\phi(b) = f(b)$$ [obtained by putting $x = a$ and $x = b$ in (1)] ...(3)

Using (3) in (2), we have

$$f(x) = f(a) + (b-a)f'(a) + (b-a)^2 A \qquad ...(4)$$

i. As $f(x), f'(x)$, are continuous on $[a, b]$ and $(b-x)$, $(b-x)^2$ being polynomial are also continuous on $[a, b]$

∴ $\phi(x)$ is continuous on $[a, b]$

ii. As $f(x), f'(x)$, are derivable on (a, b) and $(b-x)$, $(b-x)^2$ being polynomial are also derivable on (a, b)

∴ $\phi(x)$ is derivable on (a, b)

iii. Also $\phi(a) = \phi(b)$

Thus, $\phi(x)$ satisfies all three conditions of Rolle's theorem.

Hence there exists a real number θ, $0 < \theta < 1$ such that $\phi'[a + \theta(b - a)] = 0$ ...(5)

Differentiating (1) w.r.t. x, we have

$$\phi'(x) = f'(x) + (b - x) f''(x) + (-1) f'(x) + 2(b - x)(-1) A$$

or $$\phi'(x) = (b - x) f''(x) - 2(b - x) A$$

or $$\phi'(x) = (b - x) [f''(x) - 2A]$$

Putting $x = a + \theta\,(b - a)$, we have

$$\phi'[a + \theta(b - a)] = [b - a - \theta(b - a)] [f''(a + \theta(b - a) - 2A]$$

$$\Rightarrow \quad 0 = (b - a)(1 - \theta) [f''(a + \theta\,(b - a) - 2A]$$

[from (5) and as $b - a \neq 0$, $1 - \theta \neq 0$]

$$\Rightarrow \quad A = \frac{1}{2} f''(a + \theta\,(b - a))$$

Putting the value of 'A' in (4), we have

$$f(b) = f(a) + (b - a) f'(a) + \frac{(b-a)^2}{2!} f''(a + \theta(b - a))$$

Example 2.19. *Show that* $\log(x + h) = \log x + \dfrac{h}{x} - \dfrac{h^2}{2x^2} + \ldots + (-1)^{n-1} \dfrac{h^n}{n(x+\theta h)^n}$

Solution. Let $f(x + h) = \log(x + h)$

$\therefore$ $$f(x) = \log x$$

$$f'(x) = \frac{1}{x}$$

$$f''(x) = -\frac{1}{x^2}$$

$$f''' = \frac{2}{x^3}$$

$$f^{iv}(x) = \frac{-6}{x^4} = (-1)^3 \cdot \frac{3!}{x^4}$$

....

....

Continuing like this

$$f^{n-1}(x) = (-1)^{n-2} \frac{(n-2)!}{x^{n-1}}$$

$$f^{n}(x) = (-1)^{n-1} \frac{(n-1)!}{x^{n}}$$

$$f^{n}(x + \theta h) = \frac{(-1)^{n-1}(n-1)!}{(x+\theta h)^n}$$

By Talylor's theorem with Lagrange's form of remainder, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \dots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(x) + \frac{h^n}{n!} f^n(x+\theta h)$$

After putting all values, we have

$$\log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3!} \times \frac{2!}{x^3} + \dots + \frac{h^{n-1}}{(n-1)!} \frac{(-1)^{n-2}(n-2)!}{x^{n-1}} + \frac{h^n}{n!} \frac{(-1)^{n-1}(n-1)!}{(x+\theta h)^n}$$

or $$\log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} + \dots + \frac{h^{n-1}(-1)^{n-2}}{(n-1)\,x^{n-1}} + \frac{(-1)^{n-1}h^n}{n(x+\theta h)^n}$$

Example 2.20. *Expand $e^{ax} \sin bx$ by Maclaurin's theorem with Lagrange's form of remainder after n terms.*

Solution. Let $f(x) = e^{ax} \sin bx$

$\therefore$

$$f'(x) = e^{ax} \cos bx \,.\, b + ae^{ax} \sin bx$$

$$= e^{ax}(b \cos bx + a \sin bx)$$

$$f''(x) = e^{ax}(-b^2 \sin bx + ab \cos bx) + ae^{ax}(b \cos bx + a \sin bx)$$

$$= e^{ax}[(a^2 - b^2) \sin bx + 2ab \cos bx]$$

$$f'''(x) = e^{ax}[(a^2 - b^2) \cos bx \,.\, b - 2ab \sin bx \,.\, b + ae^{ax}[(a^2 - b^2) \sin bx + 2ab \cos bx]$$

$$= e^{ax}[b(3a^2 - b^2) \cos bx + (a^3 - 3ab^2) \sin bx]$$

Continuing like this, we have

$$f^n(x) = (a^2 + b^2)^{n/2} e^{ax} \sin\left(bx + n \tan^{-1}\frac{b}{a}\right)$$

At $x = 0$, we have

$$f(0) = 0, f'(0) = b, \qquad f'(0) = 2ab,$$

$$f'''(0) = b(3a^2 - b^2)$$

$$f^n(\theta x) = (a^2 + b^2)^{n/2} e^{a\theta x} \sin\left(b\theta x + n \tan^{-1}\frac{b}{a}\right)$$

According to Maclaurin's theorem with Lagrange's form of remainder, we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^{n-1}}{(n-1)} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x)$$

After putting all values, we have

$$e^{ax} \sin bx = 0 + x \,.\, b + \frac{x^2}{2!}(2ab) + \dots + \frac{x^n}{n!}(a^2 + b^2)^{n/2} e^{a\theta x} \sin\left(b\theta x + n \tan^{-1}\frac{b}{a}\right)$$

or $$e^{ax} \sin bx = bx + \frac{x^2}{2!}(2ab) + \frac{x^3}{3!} b(3a^2 - b^2) + \dots + \frac{x^n}{n!}(a^2 + b^2)^{n/2} e^{a\theta x} \sin\left(b\theta x + n \tan^{-1}\frac{b}{a}\right)$$

EXERCISE 2.4

1. Show that for every value of x, the expansion,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + (-1)^n \frac{x^{2n}}{(2n)!}\sin\theta x,\ 0 < \theta < 1.$$

2. With the help of Maclaurin's expansion, show that

   a. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-2}\frac{x^{n-1}}{n-1} + (-1)^{n-1}\frac{x^n}{n(1+\theta x)^n}$.

   b. $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots - \frac{x^{n-1}}{n-1} - \frac{x^n}{n(1-\theta x)^n}$.

3. If f' is continuous on $[a, a+h]$ and derivable on $(a, a+h)$, then prove that there exist a real number c between a and $(a+h)$, such that $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(c)$.

4. The expansion of a function $f(x) = (1-x)^{5/2}$ is given by $f(0) + xf'(0) + \frac{x^2}{2!}f''(\theta x)$. Find the value of θ as $x \to 1$.

5. Expand $\sqrt{x}$ in ascending power of x by using Maclaurin's theorem, if possible.

6. Expand the function $f(x) = a^x$ by using Maclaurin's theorem with Lagranges form of remainder after n terms.

7. Expand $e^{ax}\sin bx$ by using Maclaurin's theorem with Cauchy's form of remainder after n terms.

Answers

4. $\theta = \frac{9}{25}$

6. $1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \dots + \frac{x^{n-1}}{(n-1)!}(\log a)^{n-1} + \frac{x^n}{n!}a^{\theta x}(\log a)^n$

7. $bx + (2ab)\frac{x^2}{2!} + b(3a^2 - b^2)\frac{x^3}{3!} + \dots + (a^2+b^2)^{n/2}(1-\theta)^{n-1}\frac{x^n}{(n-1)!}e^{a\theta x}\sin\left(b\theta x + n\tan^{-1}\frac{b}{a}\right)$

INTERESTING FACTS

- It is even used in signal processing industry where we need to approximate sinusoidal functions.
- It is used in transistors and amplifiers industry to check the effect of signal.

VIDEO REFERENCES

Taylor's Theorem 1

Taylor's Theorem 2

APPLICATIONS TO REAL LIFE

- These help in calculating the approximate values of many functions on computers and calculators.
- They are very useful in solving the limits and determining several infinite sums.
- These are very helpful in understanding the asymptotic behaviour of functions.

2.3 INDETERMINATE FORMS AND L′HOSPITAL'S RULE

Let $f(x)$ and $g(x)$ be the given two functions. Then the limit of $f(x)/g(x)$ as $x \to c$ is, in general, equal to the limit of the numerator divided by the limit of the denominator. But when those two limits are both zero, the quotient reduces to the form 0/0.

The form 0/0 is called an indeterminate form.

Mathematically, it can be expressed as,

For evaluating the limit, $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{l}{m}$

if $l = 0$, $m \neq 0$, then $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = 0$

if $l \neq 0$, $m = 0$, then $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \infty$

if $l = 0$, $m = 0$, then $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{0}{0}$

cannot be evaluated and this is called indeterminate form.

Different indeterminate forms are represented by symbols as follows

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 0^0, 1^\infty, \infty^0$$

Here, we will explain all these indeterminate forms with examples.

2.3.1 L'Hospital Rule for Evaluation of Indeterminate form $\frac{0}{0}$ (Type-I)

Theorem: Let the functions f and g are differentiable function at $x = a$ and $f(a) = 0 = g(a)$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Proof: As $f(a) = 0 = g(a)$

We can write, $\dfrac{f(x)}{g(x)} = \dfrac{f(x) - f(a)}{g(x) - g(a)}$

Dividing by $x - a$, we have

$$\frac{f(x)}{g(x)} = \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}}$$

Taking limit on both sides, we have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}}$$

$$= \frac{\displaystyle\lim_{x \to a} \frac{f(x)-f(a)}{x-a}}{\displaystyle\lim_{x \to a} \frac{g(x)-g(a)}{x-a}} \qquad \left[\because \lim_{x \to b} \frac{f(x)}{g(x)} = \frac{\displaystyle\lim_{x \to b} f(x)}{\displaystyle\lim_{x \to a} g(x)}\right]$$

$$= \frac{f'(a)}{g'(a)} \qquad \text{[As per definition of differentiability]}$$

$$= \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Generally, if $f(a) = f'(a) = f''(a) \ldots = f^{n-1}(a) = 0$

$g(a) = g'(a) = g''(a) \ldots = g^{n-1}(a) = 0$

and $g^n(a) \neq 0$

then, if $\lim_{x \to a} \frac{f^n(x)}{g^n(x)}$ exists,

then $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^n(x)}{g^n(x)}$$

This is known as L'Hospital's Rule.

Working Rule

If $\lim_{x \to a} \frac{f(x)}{g(x)}$ is undefined and is of form $\frac{0}{0}$, then evaluating the limit by following procedures:

1. Differentiate the numerator and denominator separately *i.e.*, apply L'Hospital's rule

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Two cases arise:

Case I: If $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ is not of the form $\frac{0}{0}$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Case II: If $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ is of the form $\frac{0}{0}$, then again differentiate numerator and denominator separately *i.e.*, apply L'Hospital Rule such that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)}$$

2. Repeat the above procedure (Case-II) till $\lim_{x \to a} \frac{f^n(x)}{g^n(x)}$ reach to determinate form, *i.e.*,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^n(x)}{g^n(x)}$$

SOME SOLVED EXAMPLES (TYPE-I)

Example 2.21. *Evaluate* $\lim_{x \to 0} \frac{1 - \cos x}{3x^2}$.

Solution. Given, $\lim_{x \to 0} \frac{1 - \cos x}{3x^2}$ $\left[\frac{0}{0} \text{ form}\right]$

Apply L'Hospital Rule, $= \lim_{x \to 0} \frac{\sin x}{6x}$ $\left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{x \to 0} \frac{\cos x}{6}$$

$$= \frac{1}{6}$$

Example 2.22. *Evaluate* $\lim_{x \to 1} \frac{x^x - x}{x - 1 - \log x}$.

Solution. Given, $\lim_{x \to 1} \frac{x^x - x}{x - 1 - \log x}$ $\left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{x \to 1} \frac{x^x(1 + \log x) - 1}{1 - \frac{1}{x}} \quad \text{(Apply L'Hospital Rule)}$$

$$= \lim_{x \to 1} \frac{x^x + x^x \log x - 1}{1 - \frac{1}{x}} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 1} \frac{x^x(1 + \log x) + x^x \cdot \frac{1}{x} + x^x(1 + \log x)\log x}{\frac{1}{x^2}}$$

$$= \lim_{x \to 1} \frac{x^x + x^x \log x + x^{x-1} + (x^x + x^x \log x)\log x}{\frac{1}{x^2}}$$

$$= \frac{1 + 0 + 1 + 0}{1} = 2.$$

Example 2.23. *Find the value of 'a' and 'b' so that* $\lim\limits_{x\to 0}\dfrac{x(1+a\cos x)-b\sin x}{x^3}=1$.

Solution. Given, $\lim\limits_{x\to 0}\dfrac{x(1+a\cos x)-b\sin x}{x^3} = 1$ $\left[\dfrac{0}{0}\text{ form}\right]$

$$\Rightarrow \quad \lim_{x\to 0}\frac{x(-a\sin x)+(1+a\cos x)-b\cos x}{3x^2} = 1 \quad \text{...(1)}$$

Since R.H.S. of (1) is finite, so L.H.S. must finite, when $x\to 0$

But denominator $\to 0$ as $x\to 0$

and numerator $\to 0$ as $x\to 0$

$$\Rightarrow \quad 1 + a - b = 0$$

$$\Rightarrow \quad a - b = -1 \quad \text{...(2)}$$

Again Applying L'Hospital Rule on L.H.S. of (1), we have

$$\lim_{x\to 0}\frac{-a\sin x - ax\cos x - a\sin x + b\sin x}{6x} = 1 \quad \left[\frac{0}{0}\text{ form}\right]$$

$$\lim_{x\to 0}\frac{-a\cos x - a\cos x + ax\sin x - a\cos x + b\cos x}{6} = 1$$

$$\Rightarrow \quad \frac{-3a+b}{6} = 1 \quad \Rightarrow \quad -3a + b = 6 \quad \text{...(3)}$$

Using (2) and (3), we get, $a = -5/2$, $b = -3/2$ **Answer**

Example 2.24. *Evaluate* $\lim\limits_{x\to 0}\dfrac{e^x\sin x - x - x^2}{x^3}$.

Solution. $\lim\limits_{x\to 0}\dfrac{e^x\sin x - x - x^2}{x^3}$ $\left[\dfrac{0}{0}\text{ form}\right]$

Applying L'Hospital Rule,

$$\lim_{x\to 0}\frac{e^x\cos x + e^x\sin x - 1 - 2x}{3x^2}$$

or

$$\lim_{x\to 0}\frac{e^x(\cos x + \sin x) - 1 - 2x}{3x^2} \quad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{e^x[\cos x + \sin x] + e^x[-\sin x + \cos x] - 2}{6x}$$

$$= \lim_{x\to 0}\frac{2e^x\cos x - 2}{6x} \quad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{2e^x\cos x - 2e^x\sin x}{6} = \frac{2}{6} = \frac{1}{3} \quad \textbf{Answer}$$

Example 2.25. *Find 'a' such that* $\lim\limits_{x\to 0}\dfrac{a\sin x - \sin 2x}{\tan^3 x}$ *is finite.*

Solution. $\lim\limits_{x\to 0}\dfrac{a\sin x - \sin 2x}{\tan^3 x}$ $\left[\dfrac{0}{0}\text{ form}\right]$

Applying L'Hospital Rule, we have

$$\lim_{x\to 0} \frac{a\cos x - 2\cos 2x}{3\tan^2 x \,.\, \sec^2 x} = \frac{a-2}{0}$$

But it is given that, $\lim_{x\to 0} \dfrac{a\sin x - \sin 2x}{\tan^3 x}$ is finite

$\therefore$ $\qquad a - 2 = 0$

$\Rightarrow$ $\qquad a = 2$ **Answer**

2.3.1.1 Evaluation of Limit by Method of Expansion of Series

i. $a^x = 1 + x\log a + \dfrac{x^2}{2!}(\log a)^2 + \dfrac{x^3}{3!}(\log a)^3 + \ldots\ldots$

ii. $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\ldots$

iii. $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots\ldots, |x| < 1$

iv. $\log(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} \ldots\ldots, |x| < 1$

v. $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots\ldots, |x| < 1$

vi. $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots\ldots, |x| < 1$

vii. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots\ldots, \quad \forall x$

viii. $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots\ldots, \quad \forall x$

ix. $\sin hx = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots\ldots, \quad \forall x$

x. $\cos hx = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots\ldots, \quad \forall x$

xi. $\tan x = x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 + \ldots\ldots, \quad \forall x$

xii. $(1+x)^n = 1 + nx + \dfrac{n(n-1)x^2}{2!} + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots$

Example 2.26. *Evaluate* $\lim_{x\to 0} \dfrac{1-\cos x}{x\log(1+x)}$.

Solution. Here $\lim_{x\to 0} \dfrac{1-\cos x}{x\log(1+x)}$ $\qquad \left[\dfrac{0}{0}\text{ form}\right]$

$$= \lim_{x \to 0} \frac{1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots\ldots\right)}{x\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\right)}$$

$$= \lim_{x \to 0} \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \ldots\ldots}{x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \ldots\ldots}$$

$$= \lim_{x \to 0} \frac{x^2\left(\frac{1}{2!} - \frac{x^2}{4!} + \ldots\right)}{x^2\left(1 - \frac{x}{2} + \frac{x^2}{3} \ldots\right)}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2} - \frac{x^2}{4!} + \ldots}{1 - \frac{x}{2} + \frac{x^2}{3} \ldots}$$

$$= \frac{\frac{1}{2}}{1} = \frac{1}{2} \quad \textbf{Answer}$$

Alternate Method: $\lim_{x \to 0} \frac{1 - \cos x}{x \log(1 + x)}$ $\left[\frac{0}{0} \text{ form}\right]$

Apply L'Hospital Rule, $\lim_{x \to 0} \frac{\sin x}{\frac{x}{1+x} + \log(1+x)}$ $\left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{x \to 0} \frac{\cos x}{\frac{1}{(1+x)^2} + \frac{1}{1+x}} = \frac{1}{1+1} = \frac{1}{2}.$$

Example 2.27. *Evaluate* $\lim_{x \to 0} \frac{x \cos x - \log(1 + x)}{x^2}$.

Solution. Given, $\lim_{x \to 0} \frac{x \cos x - \log(1 + x)}{x^2}$ $\left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{x \to 0} \frac{x\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots\right] - \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right]}{x^2}$$

$$= \lim_{x \to 0} \frac{\left[x - \frac{x^3}{2!} + \frac{x^5}{4!} \ldots\right] - \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right]}{x^2}$$

$$= \lim_{x \to 0} \frac{\left[\frac{x^2}{2} - \frac{5x^3}{6} + \dots\right]}{x^2}$$

$$= \lim_{x \to 0} \left[\frac{1}{2} - \frac{5x}{6}\right] = \frac{1}{2} \quad \textbf{Answer}$$

EXERCISE 2.5

1. Evaluate the following:

a. $\lim_{x \to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}$

b. $\lim_{x \to 0} \frac{1 - \cos x^2}{x^2 \sin x^2}$

c. $\lim_{x \to 0} \frac{1 + \sin x - \cos x + \log(1-x)}{x \tan^2 x}$

d. $\lim_{x \to 0} \frac{(\tan^{-1} x)^2}{\log(1+x^2)}$

e. $\lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x^2 \sin x}$

f. $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$

g. $\lim_{x \to 0} \frac{a^x - 1}{b^x - 1}$

h. $\lim_{x \to 0} \frac{3^x - 2^x}{\sqrt{x}}$

i. $\lim_{x \to 0} \frac{\sin hx - x}{\sin x - x\cos x}$

j. $\lim_{x \to 0} \frac{x - \sin x}{e^{\sin x} - e^x}$

2. Evaluate the following limits:

a. $\lim_{a \to b} \frac{a^b - b^a}{a^a - b^b}$

b. $\lim_{x \to 0} \frac{\cos hx - \cos x}{x \sin x}$

c. $\lim_{x \to b} \frac{x^b - b^x}{x^x - b^b}$

d. $\lim_{x \to 0} \frac{\tan x - x}{x^2 \tan x}$

e. $\lim_{x \to 0} \frac{xe^x - \log(1+x)}{x^2}$

f. $\lim_{x \to \pi/2} \frac{\left(\frac{\pi}{2} - x\right)^2 \sin x}{\cos^2 x}$

g. $\lim_{x \to 0} \frac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}$

h. $\lim_{x \to 0} \frac{e^x - 2\cos x + e^{-x}}{x \sin x}$

3. If $\lim_{x \to 0} \frac{re^x - q\cos x + pe^{-x}}{x \tan x} = 3$, find the values of p, q and r.

4. Find the value of a, b, c so that $\lim_{x \to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2$.

5. Evaluate:

i. $\lim_{x \to 0} \frac{\sin x}{x}$

ii. $\lim_{x \to 0} \frac{\tan x}{x}$

Answers

1.	a. 1	b. $\frac{1}{2}$	c. $-\frac{1}{2}$	d. 1
	e. $\frac{1}{3}$	f. 1	g. $\frac{\log a}{\log b}$	h. 0
	i. $\frac{1}{2}$	j. –1		
2.	a. $\frac{1-\log b}{1+\log b}$	b. $\frac{1}{2}$	c. $\frac{1-\log b}{1+\log b}$	d. $\frac{1}{3}$
	e. 3/2	f. 1	g. –2/3	h. 2
3.	$p=\frac{3}{2}, q=3, r=\frac{3}{2}$	**4.** $a=1, b=2, c=1$	**5.** i. 1	ii. 1

2.3.2 L'Hospital Rule for Evaluation of Indeterminate Form $\frac{\infty}{\infty}$ (Type-II)

Theorem: If $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=\infty$, then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$ provided that R.H.S. exists (whether finite or infinite).

Working Rule:

1. For evaluating the indeterminate form $\frac{\infty}{\infty}$, change them to the form $\frac{0}{0}$ and then solve.
2. When $x\to\infty$, then $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}$ when $x\to\infty$, change $x\to\frac{1}{y}$ so that $y\to 0$

   Let $x=\frac{1}{y}$ then $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{y\to 0}\frac{f\left(\frac{1}{y}\right)}{g\left(\frac{1}{y}\right)}$ and proceed further.

SOME SOLVED EXAMPLES (TYPE-II)

Example 2.28. *Evaluate* $\lim_{x\to 0^+}\frac{\log\tan 2x}{\log\tan x}$.

Solution. Given, $\lim_{x\to 0^+}\frac{\log\tan 2x}{\log\tan x}$ $\left[\frac{\infty}{\infty}\text{ form}\right]$

$$=\lim_{x\to 0^+}\frac{\frac{1}{\tan 2x}\times\sec^2 2x\times 2}{\frac{1}{\tan x}\times\sec^2 x}$$

$$= \lim_{x \to 0^+} \frac{2 \tan x \,.\, \sec^2 2x}{\tan 2x \,.\, \sec^2 x}$$

$$= \lim_{x \to 0^+} \frac{2 \dfrac{\sin x}{\cos x} \times \dfrac{1}{\cos^2 2x}}{\dfrac{\sin 2x}{\cos 2x} \times \dfrac{1}{\cos^2 x}}$$

$$= \lim_{x \to 0^+} \frac{2 \sin x}{\sin 2x} \frac{\cos x}{\cos 2x}$$

$$= \lim_{x \to 0^+} \frac{\sin 2x}{\sin 2x \cos 2x} = \lim_{x \to 0^+} \frac{1}{\cos 2x}$$

$$= 1$$

Example 2.29. *Evaluate* $\lim_{x \to \infty} \dfrac{2x^4 + 3x^3 - 100}{4x^4 + x^2 + 2x + 100}$.

Solution. Given $\lim_{x \to \infty} \dfrac{2x^4 + 3x^3 - 100}{4x^4 + x^2 + 2x + 100}$

Put $x = \dfrac{1}{y}$ as $x \to \infty \quad \Rightarrow \quad y \to 0$

$$\therefore \qquad \lim_{y \to 0} \frac{2 \,.\, \dfrac{1}{y^4} + 3 \,.\, \dfrac{1}{y^3} - 100}{4 \,.\, \dfrac{1}{y^4} + \dfrac{1}{y^2} + 2 \,.\, \dfrac{1}{y} + 100}$$

$$= \lim_{y \to 0} \frac{2 + 3y - 100y^4}{4 + y^2 + 2y^3 + 100y^4} = \frac{2}{4} = \frac{1}{2}$$

Example 2.30. *Evaluate* $\lim_{x \to \pi/2} \dfrac{\tan x}{\tan 3x}$.

Solution. Here $\lim_{x \to \pi/2} \dfrac{\tan x}{\tan 3x}$ $\qquad \left[\dfrac{\infty}{\infty} \text{ form}\right]$

$$\Rightarrow \qquad \lim_{x \to \pi/2} \frac{\sec^2 x}{3 \sec^2 3x}$$

$$= \lim_{x \to \pi/2} \frac{1}{3} \frac{\cos^2 3x}{\cos^2 x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to \pi/2} \frac{1}{3} \left[\frac{-2 \cos 3x \,.\, \sin 3x \,.\, 3}{-2 \cos x \,.\, \sin x}\right]$$

$$= \lim_{x \to \pi/2} \frac{\sin 6x}{\sin 2x} \qquad [\because \quad \sin 2x = 2 \sin x \cos x] \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to \pi/2} \frac{6 \cos 6x}{2 \cos 2x} = \frac{3 \cos 3\pi}{\cos \pi} = \frac{3(-1)}{(-1)} = 3.$$

2.3.3 L' Hospital Rule for Evaluation of Indeterminate Form 0 × ∞ (Type-III)

If $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = \infty$, then $\lim\limits_{x \to a} f(x) \,.\, g(x)$ is of the form $0 \times \infty$

Converting,

$$\lim_{x \to a} f(x) \,.\, g(x) = \lim_{x \to a} \frac{f(x)}{\frac{1}{g(x)}} \text{ or } \lim_{x \to a} \frac{g(x)}{\frac{1}{f(x)}}$$

which is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ respectively and then solve with previous discussed methods.

SOME SOLVED EXAMPLES (TYPE-III)

Example 2.31. *Evaluate* $\lim\limits_{x \to 0^+} x \log x.$

Solution. Here, $\lim\limits_{x \to 0^+} x \log x$ [$0 \times \infty$ form]

$$= \lim_{x \to 0^+} \frac{\log x}{1/x} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

Applying L'Hospital Rule

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x \to 0^+} (-x) = 0$$

Example 2.32. *Evaluate* $\lim\limits_{x \to 0} x \tan\left(\frac{\pi}{2} - x\right).$

Solution. Given $\lim\limits_{x \to 0} x \cot x$ [$0 \times \infty$ form]

$$= \lim_{x \to 0} \frac{x}{\frac{1}{\cot x}} = \lim_{x \to 0} \frac{x}{\tan x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{1}{\sec^2 x} = 1$$

Example 2.33. *Evaluate* $\lim\limits_{x \to \frac{\pi^-}{2}} \left(x - \frac{\pi}{2}\right) \tan x.$

Solution. Given $\lim\limits_{x \to \frac{\pi^-}{2}} \left(x - \frac{\pi}{2}\right) \tan x$ [$0 \times \infty$ form]

$$= \lim_{x\to \frac{\pi^-}{2}} \frac{\left(x-\frac{\pi}{2}\right)}{\frac{1}{\tan x}} = \lim_{x\to \frac{\pi^-}{2}} \frac{\left(x-\frac{\pi}{2}\right)}{\cot x} \qquad \left[\frac{0}{0}\text{ form}\right]$$

Apply L'Hospital Rule,

$$\lim_{x\to \frac{\pi^-}{2}} \frac{1}{-\text{cosec}^2 x} = -1$$

Example 2.34. *Evaluate* $\lim_{x\to\infty} 2^x \sin \frac{a}{2^x}$.

Solution. Given, $\lim_{x\to\infty} 2^x \sin\frac{a}{2^x}$ $\qquad [0\times\infty \text{ form}]$

$$\Rightarrow \qquad \lim_{x\to\infty} \frac{\sin\frac{a}{2^x}}{\frac{1}{2^x}} \qquad \left[\frac{0}{0}\text{ form}\right]$$

Applying L'Hospital Rule

$$\Rightarrow \qquad \lim_{x\to\infty} \frac{\cos\frac{a}{2x}\left(\frac{-a\,.\,2^x\log 2}{(2^x)^2}\right)}{\frac{-2^x\log 2}{(2^x)^2}}$$

$$= \lim_{x\to\infty} a\cos\frac{a}{2^x} = a$$

2.3.4 L'Hospital Rule for Evaluation of the Indeterminate Form $\infty - \infty$ (Type-IV)

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} (f(x) - g(x))$ is of the form $\infty - \infty$

In this form, convert,

$$\lim_{x\to a} (f(x)-g(x)) = \lim_{x\to a} \frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)\,g(x)}} \qquad \left[\frac{0}{0}\text{ form}\right]$$

which can be evaluated by L'Hospital Rule as did earlier.

SOME SOLVED EXAMPLES (TYPE-IV)

Example 2.35. *Evaluate* $\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{\text{cosec } x}{x}\right)$.

Solution. Given, $\lim_{x \to 0}\left(\frac{1}{x^2} - \frac{1}{x \sin x}\right)$ $[\infty - \infty \text{ form}]$

$$= \lim_{x \to 0}\left(\frac{\sin x - x}{x^2 \sin x}\right) \qquad \left[\frac{0}{0} \text{ form}\right]$$

Applying L'Hospital Rule

$$= \lim_{x \to 0}\left(\frac{\cos x - 1}{x^2 \cos x + 2x \sin x}\right) \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0}\left(\frac{-\sin x}{-x^2 \sin x + 2x \cos x + 2x \cos x + 2 \sin x}\right)$$

$$= \lim_{x \to 0}\left(\frac{-\sin x}{-x^2 \sin x + 4x \cos x + 2 \sin x}\right) \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{-\cos x}{-x^2 \cos x - 6x \sin x + 6 \cos x} = \frac{-1}{6}$$

Example 2.36. *Evaluate* $\lim_{x \to \pi/2} (\sec x - \tan x)$.

Solution. Here, $\lim_{x \to \pi/2} (\sec x - \tan x)$ $[\infty - \infty \text{ form}]$

$$= \lim_{x \to \pi/2}\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$$

$$= \lim_{x \to \pi/2}\left(\frac{1 - \sin x}{\cos x}\right) \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to \pi/2}\left(\frac{-\cos x}{-\sin x}\right)$$

$$= \frac{\cos \frac{\pi}{2}}{\sin \frac{\pi}{2}} = \frac{0}{1} = 0$$

Example 2.37. *Evaluate* $\lim_{x \to 0}\left(\operatorname{cosec}^2 x - \frac{1}{x^2}\right)$.

Solution. Given, $\lim_{x \to 0}\left(\operatorname{cosec}^2 x - \frac{1}{x^2}\right)$

or $\lim_{x \to 0}\left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right)$ $[\infty - \infty \text{ form}]$

$$= \lim_{x \to 0}\left(\frac{x^2 - \sin^2 x}{x^2 \sin^2 x}\right) \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{x^2-\sin^2 x}{x^4}\right)\left(\frac{x^2}{\sin^2 x}\right)$$

$$= \lim_{x\to 0}\left(\frac{x^2-\sin^2 x}{x^4}\right)\lim_{x\to 0}\left(\frac{x^2}{\sin^2 x}\right)$$

$$= \lim_{x\to 0}\left(\frac{x^2-\sin^2 x}{x^4}\right) \qquad \left(\because \lim_{x\to 0}\frac{\sin x}{x}=1\right)\left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{x^2-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\dots\right)^2}{x^4}$$

$$= \lim_{x\to 0}\frac{x^2-x^2-\frac{x^4}{3}+\frac{x^6}{60}-\dots}{x^4} = \lim_{x\to 0}\frac{\frac{x^4}{3}-\frac{x^6}{60}\dots}{x^4}$$

$$= \lim_{x\to 0}\frac{1}{3}-\frac{x^2}{60} + \text{term containing higher power of } x$$

$$= \frac{1}{3}.$$

EXERCISE 2.6

1. Evaluate the following limits:

a. $\lim_{x\to 0^+}\frac{\log\sin x}{\cot x}$

b. $\lim_{x\to a^+}\frac{\log(x-a)}{\log(e^x-e^a)}$

c. $\lim_{x\to\infty}\frac{x^3-8x^2+2x+1}{x^4-x^2+2x-3}$

d. $\lim_{x\to\infty}\frac{\log x}{x}$

e. $\lim_{x\to 0^+}\frac{\operatorname{cosec} x}{\log x}$

f. $\lim_{x\to 0^+}\frac{\log x}{\cot x}$

g. $\lim_{x\to\infty}\frac{x^n}{e^x},\ n\in N$

h. $\lim_{x\to 5}\frac{\log(1-x)}{\cot(\pi x)}$

i. $\lim_{x\to 0^+}\log_x \sin x$

2. Evaluate the following indeterminate forms:

a. $\lim_{x\to 0} x\log x\tan x$

b. $\lim_{x\to\infty} x\tan\frac{1}{x}$

c. $\lim_{x\to a}(a-x)\tan\frac{\pi x}{2a}$

d. $\lim_{x\to 1}\sec\frac{\pi}{2x}\log x$

e. $\lim_{x\to \pi/2}(1-\sin x)\tan x$

f. $\lim_{x\to\infty}(a^{1/x}-1)x$

3. Evaluate the following:

a. $\lim_{x \to 0^+} \left(\frac{1}{x} - \operatorname{cosec} x \right)$

b. $\lim_{x \to 0^+} \frac{\cot x - \frac{1}{x}}{x}$

c. $\lim_{x \to 0} \left(\frac{1}{x^2} - \cot^2 x \right)$

d. $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\log x} \right)$

e. $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{x \tan x} \right)$

f. $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{x^2} \log(1+x) \right)$

g. $\lim_{x \to \pi/2} (2x \tan x - \pi \sec x)$

h. $\lim_{x \to 0} (\operatorname{cosec} x - \cot x)$

Answers

1. a. 0   b. 1   c. 0   d. 0
e. $-\infty$ f. 0   g. 0   h. 0
i. 1

2. a. 0   b. 1   c. $2a/\pi$ d. $2/\pi$
e. 0   f. $\log a$

3. a. 0   b. –1/–3   c. $\frac{2}{3}$ d. $\frac{1}{2}$
e. $\frac{1}{3}$ f. $\frac{1}{2}$ g. –2   h. 0

2.3.5 L'Hospital Rule for Evaluation of Indeterminate Form 0° (Type-V)

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} [f(x)]^{g(x)}$ is of the form 0°.

To solve this kind of form, let $y = \lim_{x \to a} [f(x)]^{g(x)}$

$$\therefore \qquad \log y = \lim_{x \to a} g(x) \log f(x) \qquad ...(1)$$

which is of the form $0 \times \infty$ and can be solved as previous method.
We can put,

$$\lim_{x \to a} g(x) \log f(x) = l,$$

then from (1), $\quad \log y = l \quad \Rightarrow \quad y = e^l.$

SOME SOLVED EXAMPLES (TYPE-V)

Example 2.38. *Evaluate* $\lim_{x \to 0^+} x^x$.

Solution. Let $\quad y = \lim_{x \to 0^+} x^x \qquad$ [0° form]

Taking log on both sides

$$\log y = \lim_{x \to 0^+} x \log x \qquad [0 \times \infty \text{ form}]$$

$$\log y = \lim_{x \to 0^+} \frac{\log x}{\frac{1}{x}} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

Applying L'Hospital Rule,

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}}$$

$$= \lim_{x \to 0^+} -x = 0$$

Thus $\log y = 0$

$\Rightarrow$ $y = e^0 = 1$

Hence, $\lim_{x \to 0^+} x^x = 1.$

Example 2.39. *Evaluate* $\lim_{x \to 1} (1 - x^2)^{\frac{1}{\log(1-x)}}$.

Solution. Let

$$y = \lim_{x \to 1} (1 - x^2)^{\frac{1}{\log(1-x)}} \qquad [0^\circ \text{ form}]$$

then

$$\log y = \lim_{x \to 1} \frac{1}{\log(1-x)} \log(1 - x^2) \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x \to 1} \frac{\log(1-x) + \log(1+x)}{\log(1-x)}$$

$$= \lim_{x \to 1} \left(1 + \frac{\log(1+x)}{\log(1-x)}\right)$$

$$= 1 + \lim_{x \to 1} \frac{\log(1+x)}{\log(1-x)}$$

$$= 1 + \lim_{x \to 1} \frac{\frac{1}{1+x}}{\frac{1}{1-x}} = 1 + \lim_{x \to 1} \frac{1-x}{1+x}$$

$$= 1 + \frac{0}{2} = 1$$

Thus, $\log y = 1$

$\Rightarrow$ $y = e^1 = e$

Hence, $\lim_{x \to 1} (1 - x^2)^{\frac{1}{\log(1-x)}} = e.$

2.3.6 L'Hospital Rule for Evaluation of Indeterminate Form 1^∞ (Type-VI)

If $\lim\limits_{x \to a} f(x) = 1$ and $\lim\limits_{x \to a} g(x) = \infty$, then $\lim\limits_{x \to a} [f(x)]^{g(x)}$ is of the form 1^∞.

It can be solve by taking $y = \lim\limits_{x \to a} [f(x)]^{g(x)}$

$$\therefore \quad \log y = \lim_{x \to a} g(x) \log f(x) \qquad \text{...(1) } [0 \times \infty \text{ form}]$$

which can be evaluated as earlier method.

After that, let $\log y = l$ [From (1)]

then $y = e^l$

means $\lim\limits_{x \to a} [f(x)]^{g(x)} = e^l$

SOME SOLVED EXAMPLES (TYPE-VI)

Example 2.40. *Evaluate* $\lim\limits_{x \to 0} (\cos x)^{1/x^2}$.

Solution. Let
$$y = \lim_{x \to 0} (\cos x)^{1/x^2} \qquad [1^\infty \text{ form}]$$

$$\therefore \quad \log y = \lim_{x \to 0} \frac{1}{x^2} \log \cos x \qquad [0 \times \infty \text{ form}]$$

Applying L'Hospital Rule

$$= \lim_{x \to 0} \frac{\frac{1}{\cos x} \times (-\sin x)}{2x}$$

$$= \lim_{x \to 0} \frac{-\tan x}{2x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{-\sec^2 x}{2} = \frac{-1}{2}$$

Thus,
$$\log y = \frac{-1}{2}$$

$$\Rightarrow \quad y = e^{-1/2}$$

Hence
$$\lim_{x \to 0} (\cos x)^{1/x^2} = e^{-1/2}$$

Example 2.41. *Evaluate the given limit:* $\lim\limits_{x \to 0} (\cos x)^{\cot^2 x}$.

Solution. Let
$$y = \lim_{x \to 0} (\cos x)^{\cot^2 x} \qquad [1^\infty \text{ form}]$$

$$\therefore \quad \log y = \lim_{x \to 0} \cot^2 x \log \cos x \qquad [0 \times \infty \text{ form}]$$

$$= \lim_{x \to 0} \frac{\log \cos x}{\tan^2 x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

Applying L'Hospital's Rule,

$$= \lim_{x \to 0} \frac{\frac{1}{\cos x}.(-\sin x)}{2 \tan x . \sec^2 x}$$

$$= \lim_{x \to 0} \frac{-\tan x}{2 \tan x \sec^2 x}$$

$$= \lim_{x \to 0} \frac{-1}{2 \sec^2 x}$$

$$= -\frac{1}{2}$$

Thus, $\log y = -\frac{1}{2}$

$\Rightarrow$ $y = e^{-1/2}$

Hence, $\lim_{x \to 0} (\cos x)^{\cot^2 x} = e^{-1/2}$

2.3.7 L'Hospital Rule for Evaluation of Indeterminate form ∞^0 (Type-VII)

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} [f(x)]^{g(x)}$ is of the form ∞^0.

For solving this kind of Indeterminate form,

Let $y = \lim_{x \to a} [f(x)]^{g(x)}$

$\therefore$ $\log y = \lim_{x \to a} g(x) \log f(x)$ ...(1) $[0 \times \infty \text{ form}]$

which is evaluated as previously discussed method

After that, let $\log y = l$ [from (1)]

then $y = e^l$

$\Rightarrow$ $\lim_{x \to a} [f(x)]^{g(x)} = e^l$

SOME SOLVED EXAMPLES (TYPE-VII)

Example 2.42. *Evaluate* $\lim_{x \to 0} (\cot x)^{1/\log x}$.

Solution. Let $y = \lim_{x \to 0} (\cot x)^{\frac{1}{\log x}}$ $[\infty^0 \text{ form}]$

$\therefore \qquad \log y = \lim_{x \to 0} \frac{1}{\log x} \log \cot x \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$

Applying L'Hospital Rule

$$= \lim_{x \to 0} \frac{\frac{1}{\cot x} \times (-\text{cosec}^2\, x)}{\frac{1}{x}}$$

$$= \lim_{x \to 0} \frac{-x\, \text{cosec}^2 x}{\cot x}$$

$$= \lim_{x \to 0} \frac{-x}{\cos x \sin x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{-1}{\cos^2 x - \sin^2 x} = -1$$

Thus, $\qquad \log y = -1 \qquad \Rightarrow \qquad y = e^{-1} = \frac{1}{e}$

Hence $\qquad \lim_{x \to 0} (\cot x)^{\frac{1}{\log x}} = \frac{1}{e}$

Example 2.43. *Evaluate* $\lim_{x \to \infty} \left(1 - \frac{1}{2x}\right)^{x+1}$

Solution. Let $\qquad f(x) = \lim_{x \to \infty} \left(1 - \frac{1}{2x}\right)^{x+1}, x > 1$

$\therefore \qquad \log f(x) = \lim_{x \to \infty} (x+1) \log\left(1 - \frac{1}{2x}\right), x > 1$

$$\log f(x) = \lim_{x \to \infty} \frac{\log\left(1 - \frac{1}{2x}\right)}{\frac{1}{x+1}} \qquad \left[\frac{0}{0} \text{ form}\right]$$

Apply L'Hospital Rule

$$= \lim_{x \to \infty} \frac{\frac{1}{\left(1 - \frac{1}{2x}\right)} \cdot \frac{1}{2x^2}}{-\frac{1}{(x+1)^2}}$$

$$= \lim_{x \to \infty} \frac{\frac{2x}{(2x-1)} \cdot \frac{1}{2x^2}}{-\frac{1}{(x+1)^2}}$$

$$= \lim_{x \to \infty} \frac{(x+1)^2}{-x(2x-1)} \quad \Rightarrow \quad \lim_{x \to \infty} \frac{x^2+1+2x}{-2x^2+x}$$

$$= \lim_{x \to \infty} \frac{1+\frac{1}{x^2}+\frac{2}{x}}{-2+\frac{1}{x}} \qquad \text{[Dividing numerator and denominator by } x^2\text{]}$$

$$= -\frac{1}{2}$$

Thus, $\log f(x) = -\frac{1}{2}$

$\Rightarrow$ $f(x) = e^{-1/2}$

$\Rightarrow$ $\lim_{x \to \infty} \left(1-\frac{1}{2x}\right)^{x+1} = e^{-1/2}$

Example 2.44. *Evaluate* $\lim_{x \to \pi/2} (\sec x)^{\cot x}$.

Solution. Let $y = \lim_{x \to \pi/2} (\sec x)^{\cot x}$ $\qquad [\infty^0 \text{ form}]$

$\therefore$ $\log y = \lim_{x \to \pi/2} \cot x \log \sec x$ $\qquad [0 \times \infty \text{ form}]$

$$= \lim_{x \to \pi/2} \frac{\log \sec x}{\tan x} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

Applying L'Hospital Rule,

$$= \lim_{x \to \pi/2} \frac{\frac{1}{\sec x}.\sec x \tan x}{\sec^2 x}$$

$$= \lim_{x \to \pi/2} \frac{\tan x}{\sec^2 x}$$

$$= \lim_{x \to \pi/2} \sin x \cos x$$

$$= 0$$

Thus, $\log y = 0$

$\Rightarrow$ $y = e^0 = 1$

Hence $\lim_{x \to \pi/2} (\sec x)^{\cot x} = 1$

Example 2.45. *Prove that* $\lim_{x \to \infty} \left(\frac{2x+3}{2x+1}\right)^{3x+2} = e^3$

Solution. Let $y = \lim_{x \to \infty} \left(\frac{2x+3}{2x+1}\right)^{3x+2}$ $\qquad [1^\infty \text{ form}]$

$\therefore$ $\log y = \lim_{x \to \infty} (3x+2) \log\left(\frac{2x+3}{2x+1}\right)$

$$= \lim_{x\to\infty} (3x+2)\log\left(1+\frac{2}{2x+1}\right) \qquad [0\times\infty \text{ form}]$$

$$= \lim_{x\to\infty} (3x+2)\left[\frac{2}{2x+1}-\left(\frac{2}{2x+1}\right)^2 . \frac{1}{2}+...\right]$$

$$= \lim_{x\to\infty} \left[\frac{2(3x+2)}{(2x+1)}-\left(\frac{2}{2x+1}\right)^2 . \frac{(3x+2)}{2}+...\right]$$

$$= \lim_{x\to\infty} \left[\frac{2}{2x}\frac{\left(1+\frac{2}{3x}\right).3x}{\left(1+\frac{1}{2x}\right)}-\frac{2\,.\,3x\left(1+\frac{2}{3x}\right)}{(2x)^2\left(1+\frac{1}{2x}\right)^2}+...\right]$$

$$= \lim_{x\to\infty} \left[\frac{3\left(1+\frac{2}{3x}\right)}{\left(1+\frac{1}{2x}\right)}-\frac{3}{2x}\frac{\left(1+\frac{2}{3x}\right)}{\left(1+\frac{1}{2x}\right)^2}+...\right]$$

$$= 3$$

Thus, $\log y = 3$

$\Rightarrow$ $y = e^3$

Hence $\lim_{x\to\infty}\left(\frac{2x+3}{2x+1}\right)^{3x+2} = e^3$

Example 2.46. *Show that* $\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x} = \frac{-e}{2}$.

Solution. Let $y = (1+x)^{1/x}$

$\therefore$ $\log y = \frac{1}{x}\log(1+x)$

$$= \frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+...\right)$$

$$= 1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+...$$

$\Rightarrow$ $y = e^{1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+...}$

$$= e\,.\,e^t, \text{ where } t = -\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+...$$

$$= e\left[1+t+\frac{t^2}{2}+...\right]$$

$$= e\left[1+\left(\frac{-x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\ldots\right)+\frac{1}{2!}\left(-\frac{x}{2}+\frac{x^2}{3}-\ldots\right)^2+\ldots\ldots\right]$$

$$= e\left[1-\frac{x}{2}+\frac{11}{24}x^2+\ldots\right]$$

$$= e-\frac{ex}{2}+\frac{11}{24}ex^2+\ldots\ldots$$

Now, $$\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x} = \lim_{x\to 0}\frac{\left(e-\frac{ex}{2}+\frac{11}{24}ex^2+\ldots\right)-e}{x}$$

$$= \lim_{x\to 0} -\frac{e}{2}+\frac{11}{24}ex+\ldots\ldots$$

$$= \frac{-e}{2}$$

Example 2.47. *Evaluate* $\lim_{n\to\infty}\left[n+n^2 log\,\frac{n}{n+1}\right]$.

Solution. Given, $\lim_{n\to\infty}\left[n+n^2\log\frac{n}{n+1}\right]$

$$= \lim_{n\to\infty}\left[n+n^2\log\left(1-\frac{1}{n+1}\right)\right]$$

$$= \lim_{n\to\infty}\left[n+n^2\left(-\frac{1}{n+1}-\frac{1}{2(n+1)^2}-\frac{1}{3(n+1)^3}\ldots\right)\right]$$

$$= \lim_{n\to\infty}\left[n-\frac{n^2}{n+1}-\frac{n^2}{2(n+1)^2}-\frac{n^2}{3(n+1)^3}\ldots\right]$$

$$= \lim_{n\to\infty}\left[\frac{n}{n+1}-\frac{1}{2}\left(\frac{n}{n+1}\right)^2-\frac{1}{3}\frac{n^2}{(n+1)^3}\ldots\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{1+\frac{1}{n}}-\frac{1}{2}\left(\frac{1}{1+\frac{1}{n}}\right)^2-\frac{1}{3}\frac{1}{n\left(1+\frac{1}{n}\right)^3}\ldots\right]$$

$$= 1-\frac{1}{2}=\frac{1}{2}$$

Hence $$\lim_{n\to\infty}\left[n+n^2\log\frac{n}{n+1}\right] = \frac{1}{2}.$$

EXERCISE 2.7

1. Evaluate the following limits:

a. $\lim_{x \to \pi/2} (\cos x)^{\cos x}$ b. $\lim_{x \to 1} (x-1)^{x-1}$ c. $\lim_{x \to \infty} \left(\frac{\pi}{2} - \tan^{-1} x\right)^{1/x}$

2. Determine the following limits:

a. $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^{x}$ b. $\lim_{x \to 0} \left(\frac{1}{x}\right)^{\tan x}$ c. $\lim_{x \to \pi/2} (\sin x)^{\tan x}$ d. $\lim_{x \to 0} \left(\frac{\sin hx}{x}\right)^{1/x}$

e. $\lim_{x \to 0} (\operatorname{cosec} x)^{1/\log x}$ f. $\lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x^2}$ g. $\lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan \frac{\pi x}{2a}}$ h. $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$

i. $\lim_{x \to 0} (1 + \sin x)^{\cot x}$ j. $\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^{x}$ k. $\lim_{x \to \pi/2} (\sin x)^{\tan^2 x}$

3. a. Show that $\lim_{x \to 0^+} \frac{(1+x)^{1/x} - e + \frac{ex}{2}}{x^2} = \frac{11e}{24}$

b. Show that $\lim_{x \to 0} \frac{(1+x)^{1/x} - e + \frac{ex}{2} - \frac{11}{24}ex^2}{x^3} = \frac{-7}{16}e$

4. a. Prove that $\lim_{x \to \infty} (1+x)^{1/x} = 1$ b. Prove that $\lim_{x \to \infty} \left(1 + \frac{k}{x}\right)^{x} = e^k$

c. Prove that $\lim_{x \to \infty} (x + e^x)^{2/x} = e^2$ d. Prove that $\lim_{x \to \infty} \left(\frac{3x+1}{3x+4}\right)^{3x+2} = e^{-3}$

5. Evaluate:

a. $\lim_{x \to 0} \left(\frac{a^x + b^x}{2}\right)^{1/x}$ and then deduce $\lim_{x \to 0} \left(\frac{2^x + 3^x}{2}\right)^{1/x} = \sqrt{6}$

b. $\lim_{x \to \infty} \left(\frac{\log x}{x}\right)^{1/x}$ c. $\lim_{n \to \infty} \frac{\log(1+n^2)}{n}$

Answers

1. a. 1 b. 1 c. 1

2. a. e^a b. 1 c. 1 d. 1
e. $1/e$ f. $e^{1/3}$ g. $e^{2/\pi}$ h. $e^{-1/6}$
i. e j. e^2 k. $e^{-1/2}$

5. b. 1 c. 0

INTERESTING FACTS

- Indeterminate forms are also found in physics. We can see its usage in quantum physics, particle decay, quantum mechanics, thermodynamics etc.
- Johann Bernoulli was also engaged in the creation of this unique rule.
- L'Hospital's Rule occasionally fails by falling into a never-ending cycle.
- Although written as Hospital, but the word is pronounced as "***Hopital***".

VIDEO REFERENCES

Indeterminate Forms Part I

APPLICATIONS TO REAL LIFE

- It has a significant application in commerce domain, where continuous compounding interest rates are encountered every day especially in investments, different types of bank accounts, while paying credit cards bills, mortgages, etc.
- It is used in Gamma functions which are further used in engineering, quantum physics, statistics, astrophysics, fluid dynamics, combinatory, probability theory, etc.

2.4 MAXIMA AND MINIMA

A function f is said to have a maximum value at $x = a$ if $f(a) > f(x)$ *i.e.*, $f(x) - f(a) < 0$ for all x in a small neighbourhood of a.

A function f is said to have a minimum at $x = a$ if $f(a) < f(x)$ *i.e.*, $f(x) - f(a) > 0$ for all values of x in a small neighbourhood of a.

In the adjoining fig., $f(x)$ has a maximum value at $x = a$ since $f(a)$ is greater than the neighbouring value of $f(x)$. Similarly $f(x)$ has a minimum at $x = b$ and maximum at d.

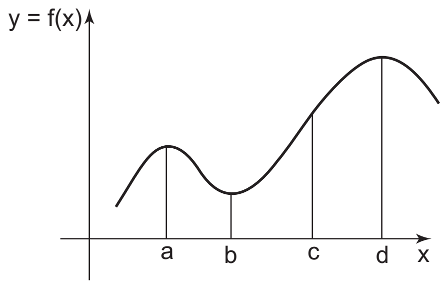

Fig. 2.7

Note that $f(x)$ has a maximum at $x = a$ even though $f(a) < f(c)$. The reason is that $f(a) > f(x)$ in a neighbourhood of a.

Thus a maximum value of $f(x)$ is not necessarily the greatest value of $f(x)$.

2.4.1 Condition for Maxima and Minima

Expanding by Taylor's theorem, about a point 'a', we have

$$f(x) = f(a) + (x-a)\,f'(a) + \frac{1}{2!}\,(x-a)^2 f''(a) + \ldots$$

i.e.,
$$f(x) - f(a) = (x-a)\,f'(a) + \frac{1}{2!}\,(x-a)^2 f''(a) + \ldots$$

or $$= (x-a)\left\{f'(a)+\frac{1}{2!}(x-a)f''(a)+\ldots\right\} \quad \ldots(1)$$

When $x - a$ is small, $f'(a)$ is numerically more than the succeeding terms. So the sign of $f(x) - f(a)$ depends upon $(x - a) f'(a)$. But this will have only one sign when $x > a$ and another when $x < a$. Therefore no maximum or minimum is possible at $x = a$ unless $f'(a) = 0$.

If $f'(a) = 0$, then (1) becomes

$$f(x)-f(a) = (x-a)^2\left\{\frac{1}{2}f''(a)+\frac{1}{6}(x-a)f'''(x)+\ldots\right\} \quad \ldots(2)$$

For small values of $x - a$, $\frac{1}{2} f''(a)$ has more numerical value than the succeeding terms. So the sign of $f(x) - f(a)$ depends on $\frac{1}{2}(x-a)^2 f''(a)$ or $f''(a)$, as $\frac{1}{2}(x-a)^2$ is always positive.

Hence the function $f(x)$ has maxima, minima at $x = a$ if

i. $f'(a) = 0$ and $f''(a) =$ negative, $f(x)$ has maximum at 'a'.
ii. $f'(a) = 0$ and $f''(a) =$ positive, $f(x)$ has minimum at a.
iii. $f'(a) = 0$ and $f''(a) = 0$, $f(x)$ has neither maximum nor minimum at $x = a$ unless $f'''(a) = 0$. The sign of $f^{iv}(a)$ will then determine the nature of $f(x)$.

2.4.2 First Derivative Test for Extrema (Maxima or Minima)

Let f be a continuous function defined on Interval $I = (a, b)$ and let c be the critical point in I *i.e.* $c \in I$, then

i. $f'(x)$ changes sign from positive to negative as x increases through c *i.e.*, $f'(x) \geq 0$ for $x \in (c - \delta, c)$ and $f'(x) \leq 0$ for $x \in (c, c + \delta)$, then c is the point of local maxima and f has local maximum at c.
ii. $f'(x)$ changes sign from negative to positive as x increases through c *i.e.*, $f'(x) \leq 0$ for $x \in (c - \delta, c)$ and $f'(x) \geq 0$ for $x \in (c, c + \delta)$, then c is the point of local minima and f has local minimum at c.
iii. $f'(x)$ does not change sign as x increases through c, then f has no extremum (neither maximum nor minimum) at c and such point is called point of inflection.

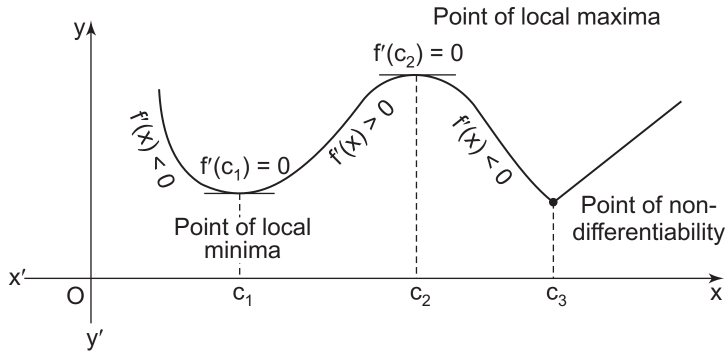

Fig. 2.8

Definition. Critical point of *f*: Let *f* be a continuous function defined on an interval I and $c \in I$, then c is called critical point if either $f'(c) = 0$ or f is not differentiable at points in I.

Definition. Let 'f' be real valued function and let c be an interior point of domain of f, then

a. c is called a point of local maxima, if there is an $h > 0$, such that $f(c) > f(x)$, $\forall\ x$ in $(c - h, c + h)$. The value $f(c)$ is called the local maximum value of f.

b. c is called a point of local minima if there is an $h > 0$ such that $f(c) < f(x)$, $\forall\ x$ in $(c - h, c + h)$.

The value $f(c)$ is called the local minimum value of f.

Geometrically, the above definition states that if $x = c$ is a point of local maxima of f, then the graph of 'f' around c will be as shown in given figure.

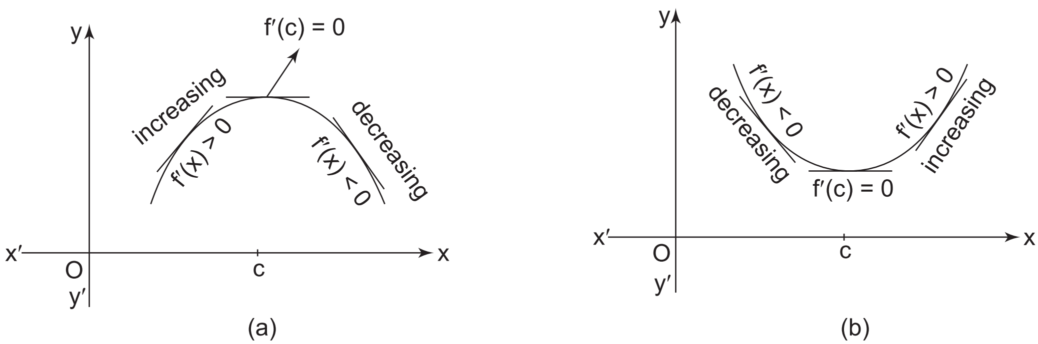

Fig. 2.9

Note that the function f is increasing (*i.e.*, $f'(x) > 0$) in the interval $(c - h, c)$ and decreasing (*i.e.*, $f'(x) < 0$) in the interval $(c, c + h)$. It suggests that $f'(c)$ must be zero.

Example 2.48. *Find all points of local maxima and minima of the function, when f is given by*

$$f(x) = x^3 - 3x + 3.$$

Solution. Given $f(x) = x^3 - 3x + 3$

Differentiating w.r.t. x, we get

$$f'(x) = 3x^2 - 3$$
$$= 3(x^2 - 1)$$
$$= 3(x - 1)(x + 1)$$

Now $f'(x) = 0 \Rightarrow 3(x - 1)(x + 1) = 0$

$\Rightarrow$ $x = 1, -1$

Thus, $x = \pm 1$ are only critical points and function can have maximum or minimum values at $x = 1, -1$.

Applying 1st derivative test

Value of x		Sign of $f'(x) = 3(x - 1)(x + 1)$
$x = 1$	→ to left (say 0.98)	$f'(x) < 0$
	→ to right (say 1.01)	$f'(x) > 0$
$x = -1$	→ to left (say –1.01)	$f'(x) > 0$
	→ to right (say –0.9)	$f'(x) < 0$

At $x = 1, f'(x)$ changes sign from negative to positive.

$\therefore$ $x = 1$ is a point of local minima, and $f(1) = (1)^3 - 3(1) + 3 = 1$ is the local minimum value.

At $x = -1, f'(x)$ changes sign from positive to negative.

$\therefore$ $x = -1$ is the point of local maxima and $f(-1) = (-1)^3 - 3(-1) + 3 = 5$ is the local maximum value.

Example 2.49. *Find all the points of local maxima and local minima of the function f, which is given by $f(x) = x^3 + 1$.*

Solution. Given $f(x) = x^3 + 1$

Differentiating w.r.t. x, we get

$$f'(x) = 3x^2$$

Now, $f'(x) = 0 \Rightarrow 3x^2 = 0 \Rightarrow x = 0$

Thus, $x = 0$ is the only critical point of 'f' and function can have maximum or minimum value at $x = 0$.

On applying 1st derivative test,

Value of x	Sign of $f'(x) = 3x^2$
$x = 0$ → to left (say –0.1)	> 0
$x = 0$ → to right (say 0.1)	> 0

Thus, at $x = 0, f'(x)$ does not changes its sign. So $x = 0$ is neither a point of local maxima nor a point of local minima.

Thus, $x = 0$ is a point of inflection.

Remark:

i. Consider a function $f(x) = x^2, x \in \mathbb{R}$

Clearly, f has minimum value at $x = 0$ and $f(0) = 0$ but f has no maximum value in $\mathbb{R}$.

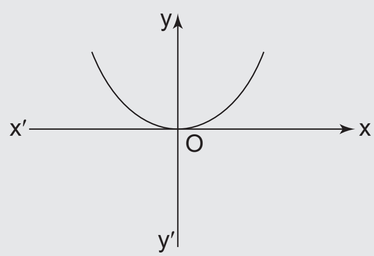

Fig. 2.10

ii. Consider a function $f(x) = |x|, x \in \mathbb{R}$

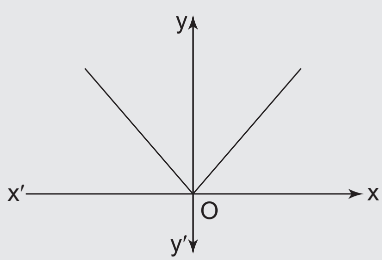

Fig. 2.11

f has a minimum value at $x = 0$. Also $f(0) = |0| = 0$, but f has no maximum value in R.

iii. Consider a function $f(x) = x, x \in (0, 2)$

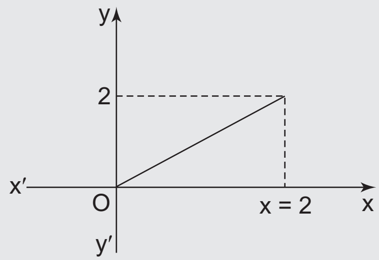

Fig. 2.12

f has neither maximum value nor minimum value in $(0, 2)$

$$f(x) = x, x \in (0, 2)\ f'(x) = 1.$$

At all points in $(0, 2)$, $f'(x) > 0$ *i.e.*, no changes in sign.

$\therefore$ f does not have maxima or minima in $(0, 2)$.

2.4.3 Second Derivative Test for Extrema (Maxima or Minima)

Let f be a twice differentiable function defined on Interval I and $c \in I$. Then

i. if $f'(c) = 0$ and $f''(c) < 0$, then $x = c$ is a point of local maxima and $f(c)$ is local maximum value of f.

ii. if $f'(c) = 0$ and $f''(c) > 0$, then $x = c$ is a point of local minima and $f(c)$ is local minimum value of f.

iii. if $f'(c) = 0$ and $f''(c) = 0$, then test fails. Further process need to be done to check for extremum values.

Example 2.50. *Find local maximum and local minimum value of the function f, given by*

$$f(x) = x^5 - 5x^4 + 5x^3 + 10, x \in \mathbb{R}.$$

Solution. Given $f(x) = x^5 - 5x^4 + 5x^3 + 10$

Now, $f'(x) = 5x^4 - 20x^3 + 15x^2$

$$= 5x^2 (x-3)(x-1)$$

$\therefore$ $f'(x) = 0 \Rightarrow 5x^2 (x-3)(x-1) = 0$

$\Rightarrow$ $x = 0, 1, 3$

Now $f''(x) = 20x^3 - 60x^2 + 30x$

$$= \begin{cases} f''(0) = 0 \\ f''(1) = -10 < 0 \\ f''(3) = 90 > 0 \end{cases}$$

By second derivative test, $x = 1$, is a point of local maxima and maximum value is

$$f(1) = 11.$$

and $x = 3$ is a point of local minima and minimum value is

$$f(3) = -17$$

At $x = 0$, second derivative test fails.

Now, by first derivative test

Value of x	Sign of $f'(x) = 5x^2(x-3)(x-1)$
$x = 0$ → left (say –0.1)	> 0
$x = 0$ → right (say 0.1)	> 0

Thus at $x = 0$, f has neither maxima nor minima.

Example 2.51. *Find local maximum or local minimum value of function f given by $f(x) = |x + 2| - 1$.*

Solution. Here, $f(x) = |x + 2| - 1$

or $$f(x) = \begin{cases} x+2-1 & x \ge -2 \\ -(x+2)-1 & x < -2 \end{cases}$$

or $$f(x) = \begin{cases} x+1 & x \ge -2 \\ -x-3 & x < -2 \end{cases}$$

Now, $$f'(x) = \begin{cases} 1 & x > -2 \\ -1 & x < -2 \end{cases}$$

By first derivative test,

Value of x	Sign of $f'(x)$
$x = -2$ → left (say –2.1)	< 0
$x = -2$ → right (say –1.9)	> 0

$f'(x)$ changes sign from negative to positive.

∴ $f(x)$ has local minima at $x = -2$ and local minimum value is

$$f(-2) = |-2 + 2| - 1 = -1$$

$f(x)$ has no local maxima and hence no maximum value.

We can also check in the following way:

Here $f(x) = |x + 2| - 1$

We know that $|x + 2| \ge 0$

For minimum, $x + 2 = 0$

⇒ $x = -2$

Thus, f has minimum value at –2, and $f(-2) = -1$.

Example 2.52. *Find maximum and minimum value of function $f(x) = \sin 2x + 5$.*

Solution. Here, $f(x) = \sin 2x + 5$

We know that $-1 \le \sin x \le 1$

∴ $-1 \le \sin 2x \le 1$

On adding (5) on both sides,

$$-1 + 5 < \sin 2x + 5 \le 1 + 5$$

$$4 \le \sin 2x + 5 \le 6$$

Thus, $f(x)$ has maximum value 6 and minimum value 4.

Example 2.53. *Find maximum and minimum value of the function* $f(x) = x\sqrt{1-x}$ *, x > 0.*

Solution. Here

$$f(x) = x\sqrt{1-x}\ , x > 0$$

$$f(x) = \sqrt{1-x} + \frac{x(-1)}{2\sqrt{1-x}}$$

$$= \sqrt{1-x} + \frac{(-x)}{2\sqrt{1-x}}$$

$$= \frac{2(1-x)-x}{2\sqrt{1-x}} = \frac{2-3x}{2\sqrt{1-x}}$$

$$f''(x) = 0 \quad \Rightarrow \quad \frac{2-3x}{2\sqrt{1-x}} = 0$$

$$\Rightarrow \qquad 2 - 3x = 0 \quad \Rightarrow \quad x = 2/3$$

Now,

$$f''(x) = \frac{1}{2}\left[\frac{\sqrt{1-x}\,(-3) - (2-3x).\dfrac{-1}{2\sqrt{1-x}}}{(1-x)}\right]$$

$$= \frac{1}{2}\left[\frac{-6\,(1-x)+(2-3x)}{2(1-x)^{3/2}}\right]$$

$$= \frac{1}{2}\left[\frac{-4+3x}{2(1-x)^{3/2}}\right]$$

At $x = 2/3$,

$$f''(2/3) = \frac{1}{2}\left(\frac{-2}{2\left(\frac{1}{3}\right)^{3/2}}\right) > 0$$

By second derivative test, $x = 2/3$ is a point of maxima and f has maximum values at $x = 2/3$

$$\therefore \qquad f(2/3) = \frac{2}{3}\sqrt{1-2/3} = \frac{2}{3}\sqrt{\frac{1}{3}}$$

$$= \frac{2}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \qquad \text{[Rationalisation]}$$

$$= \frac{2\sqrt{3}}{9}.$$

EXERCISE 2.8

1. Find maximum value of $(x-1)\,(x-2)\,(x-3)$.
2. Show that $x^3 - 3x^2 + 3x + 7$ has neither a maximum nor minimum at $x = 1$.
3. Show that $\sin x\,(1 + \cos x)$ has maximum at $x = \frac{1}{3}\,\pi$.

4. Find maximum and minimum value of following functions:

i. $f(x) = (2x - 1)^2 + 3$ ii. $f(x) = 9x^2 + 12x + 2$

iii. $f(x) = \dfrac{x^2 - 2x + 4}{x^2 + 2x + 4}$

5. Find all the points of local maxima and minima and also find the maximum and minimum values of the following functions:

i. $f(x) = x^3 - 3x$ ii. $f(x) = \sin x - \cos x, x \in (0, 2\pi)$

iii. $f(x) = \dfrac{x}{2} + \dfrac{2}{x}, x > 0$ iv. $f(x) = \dfrac{1}{x^2 + 2}$

v. $f(x) = \dfrac{x}{(1 + x^2)^2}$ vi. $f(x) = (x - 1)^4 (x - 2)^2$

vii. $f(x) = 2x^3 - 6x^2 + 6x + 5$

6. Examine whether '*f*' has local maximum or minimum at 0

i. $f(x) = \begin{cases} 2x + 3, & x > 0 \\ -3x + 1, & x \le 0 \end{cases}$ ii. $f(x) = \begin{cases} 2x + 3, & x \le 0 \\ -3x + 1, & x > 0 \end{cases}$

iii. $f(x) = \begin{cases} 2x + 3, & x \ge 0 \\ -3x + 1, & x < 0 \end{cases}$ iv. $f(x) = |x| + |x - 1|$

7. Find the maximum/minimum values of the following functions:

i. $f(x) = -|x + 1| + 3$ ii. $f(x) = |\sin 4x + 3|$

iii. $f(x) = x + 1, x \in (-1, 1)$

Answers

1. $\dfrac{2}{3\sqrt{3}}$

4. i. min. value = –2 ii. min. value = –2

iii. min. value = 1/3, max. value = 3

5. i. min at 1, $f(1) = -2$, max. at –1, $f(-1) = 2$

ii. min. at $\dfrac{7}{4}\pi, f\left(\dfrac{7}{4}\pi\right) = -\sqrt{2}$, max. at $\dfrac{3}{4}\pi, f\left(\dfrac{3}{4}\pi\right) = \sqrt{2}$

iii. min. at $x = 2, f(2) = 1$ iv. max. at $0, f(0) = \dfrac{1}{2}$

v. max. at $1, f(1) = \dfrac{1}{4}$, min. at $-1, f(-1) = -\dfrac{1}{4}$

vi. max. at $1, \dfrac{5}{3}, f(1) = 0, f\left(\dfrac{5}{3}\right) = \dfrac{16}{729}$, min. at $x = 2, f(2) = 0$

vii. neither maxima nor minima

6. i. min. ii. max.

iii. neither max. nor min. iv. minimum

7. i. max. value = 3, no min. value ii. min. value = 2, max. value = 4

iii. neither max. nor min. value

SOME MISCELLANEOUS EXAMPLES

Based on Maxima-Minima

Example 2.54. *Show that of all the rectangles with a given perimeter, the square has the largest area.*

Solution. Try yourself.

Example 2.55. *Show that of all the rectangles of a given area, the square has the smallest perimeter.*

Solution. Try yourself.

Example 2.56. *If the sum of the lengths of the hypotenuse and a side of a right-angled triangle is given, show that the area of the triangle is maximum when the angle between them is (π/3)*

Solution. Try yourself.

Example 2.57. *Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.*

Solution. Let $ABCD$ be a rectangle inscribed in a given circle with centre O and radius r.

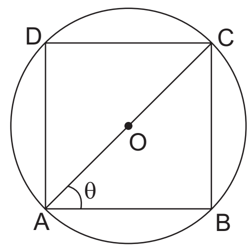

Fig. 2.13

Let $\angle CAB = \theta$.

Then, $AC = 2r$, $AB = 2r\cos\theta$ and $BC = 2r\sin\theta$.

Let A be the area of rectangle $ABCD$.

Then, $A = AB \times BC = 4r^2 \sin\theta\cos\theta = 2r^2\sin 2\theta$.

Thus, $A = 2r^2 \sin 2\theta$, where r is constant.

$$\therefore \quad \frac{dA}{d\theta} = 4r^2\cos 2\theta \text{ and } \frac{d^2A}{d\theta^2} = -8r^2\sin 2\theta.$$

Now, $$\frac{dA}{d\theta} = 0 \quad \Rightarrow \quad 4r^2\cos 2\theta = 0$$

$$\Rightarrow \quad \cos 2\theta = 0 \quad \Rightarrow \quad 2\theta = \frac{\pi}{2}, \text{ i.e., } \theta = \frac{\pi}{4}.$$

And, $$\left[\frac{d^2A}{d\theta^2}\right]_{\theta=(\pi/4)} = -8r^2\sin\frac{\pi}{2} = -8r^2 < 0.$$

$\therefore$ $\theta = (\pi/4)$ is a point of maximum.

Thus, area is maximum when $\theta = (\pi/4)$.

In this case, $$AB = 2r\cos\frac{\pi}{4} = r\sqrt{2}$$

and, $$BC = 2r\sin\frac{\pi}{4} = r\sqrt{2}$$

Thus, $AB = BC$ and therefore, $ABCD$ is a square.

Example 2.58. *Show that the triangle of maximum area that can be inscribed in a given circle is an equilateral triangle.*

Solution. Let ABC be a triangle inscribed in a given circle with centre O and radius r.

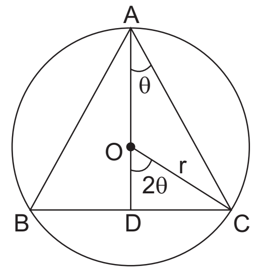

Fig. 2.14

For maximum area, the vertex A should be at a maximum distance from the base BC.

Therefore, A must lie on the diameter, perpendicular to BC. Thus, $AD \perp BC$.

So, triangle ABC must be isosceles.

Let $\angle CAD = \theta$.

Now, $BC = 2CD = 2OC \sin 2\theta = 2r \sin 2\theta$

and, $AD = (OA + OD) = (r + r\cos 2\theta)$.

Let A be the area of the triangle.

Then, $$A = \frac{1}{2} BC \times AD = r^2 \sin 2\theta\,(1 + \cos 2\theta).$$

$$\therefore \quad \frac{dA}{d\theta} = r^2 [\sin 2\theta\,(-2 \sin 2\theta) + (1 + \cos 2\theta)\,.\,2\cos 2\theta]$$

$$= r^2 [2(\cos^2 2\theta - \sin^2 2\theta) + 2\cos 2\theta] = 2r^2 [\cos 4\theta + \cos 2\theta]$$

And, $$\frac{d^2A}{d\theta^2} = 2r^2 [-4\sin 4\theta - 2\sin 2\theta] = -4r^2 (2\sin 4\theta + \sin 2\theta)$$

Now, $$\frac{dA}{d\theta} = 0 \quad \Rightarrow \quad \cos 4\theta + \cos 2\theta = 0$$

$$\Rightarrow \quad \cos 4\theta = -\cos 2\theta = \cos(\pi - 2\theta)$$

$$\Rightarrow \quad 4\theta = \pi - 2\theta \quad \Rightarrow \quad \theta = \frac{\pi}{6}.$$

and $$\left[\frac{d^2A}{d\theta^2}\right]_{\theta = (\pi/6)} = -4r^2 \left(2\sin\frac{2\pi}{3} + \sin\frac{\pi}{3}\right) = -6r^2\sqrt{3} < 0$$

$$\therefore \quad \theta = \frac{\pi}{6} \text{ is a point of maximum.}$$

So, in this case, each angle of the triangle is $(\pi/3)$.

Hence, ABC is an equilateral triangle.

Based on Volume and Area of Solids

Example 2.59. *Show that a cylinder of a given volume which is open at the top has minimum total surface area, provided its height is equal to the radius of its base.*

Solution. Try yourself.

Example 2.60. *Show that the height of a cylinder which is open at the top, having a given surface and the greatest volume, is equal to the radius of its base.*

Solution. Try yourself.

Example 2.61. *Show that the height of a cylinder which is open at the top, having a given surface area and maximum volume is* $\sin^{-1}(1/3)$.

Solution. Try yourself.

Example 2.62. *Show that the surface area of a closed cuboid with square base and given volume is minimum when it is a cube.*

Solution. Let V be the fixed volume of a closed cuboid with length a, breadth a and height h.

Let S be its surface area.

Then, $$V = (a \times a \times h) \text{ or } h = \frac{V}{a^2} \qquad ...(1)$$

Now, $$S = 2(a^2 + ah + ah) = 2(a^2 + 2ah) = 2\left(a^2 + \frac{2V}{a}\right) \qquad \text{[using (1)]}$$

i.e., $$S = 2\left(a^2 + \frac{2V}{a}\right). \therefore \frac{dS}{da} = 2\left(2a - \frac{2V}{a^2}\right) \text{ and } \frac{d^2S}{da^2} = \left(4 + \frac{8V}{a^3}\right).$$

Now, $$\frac{dS}{da} = 0 \Rightarrow V = a^3 \Rightarrow a \times a \times h = a^3 \Rightarrow h = a.$$

Now, when $h = a$, we have

$$V = a^3$$

$$\therefore \left[\frac{d^2S}{da^2}\right]_{h=a} = \left(4 + \frac{8a^3}{a^3}\right) = 12 > 0.$$

So, S is minimum when length $= a$, breadth $= a$ and height $= a$, i.e., when it is a cube.

Example 2.63. *Show that the height of a closed cylinder of given surface and maximum volume is equal to the diameter of its base.*

Solution. Try yourself.

Example 2.64. *Show that the cone of greatest volume which can be inscribed in a given sphere is such that three times its altitude is twice the diameter of the sphere. Find the volume of the largest cone inscribed in a sphere of radius R.*

Solution. Let R be the radius of the given sphere with centre O, and let V be the volume of the inscribed cone, h be its height and r be the radius of the base.

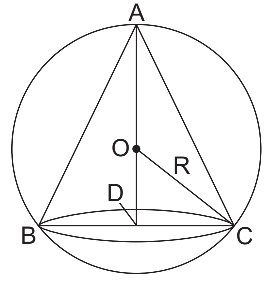

Fig. 2.15

In the given figure, we have

$$OD = AD - AO = (h - R)$$

$$\therefore R^2 = (h - R)^2 + r^2 \text{ or } r^2 = h(2R - h) \qquad ...(i)$$

Now, $$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^2 (2R - h) \qquad \text{[using (1)]}$$

$$\therefore \frac{dV}{dh} = \frac{1}{3}\pi h (4R - 3h),$$

and $$\frac{d^2V}{dh^2} = \left(\frac{4}{3}\pi R - 2\pi h\right).$$

For a maxima or minima, we have

$$\frac{dV}{dh} = 0$$

Now, $$\frac{dV}{dh} = 0 \Rightarrow \frac{1}{3}\pi h(4R - 3h) = 0$$

$$\Rightarrow \quad h = 0 \text{ or } (4R - 3h) = 0 \Rightarrow h = \frac{4}{3}R \qquad [\because \ h \neq 0]$$

and $$\left[\frac{d^2V}{dh^2}\right]_{h=(4/3)R} = -\frac{4\pi R}{3} < 0.$$

So, V is maximum when $h = \frac{4}{3}R$, *i.e.*, when $3h = 2(2R)$

i.e., 3 times the height = 2 times the diameter.

Volume of the largest cone $= \frac{1}{3}\pi \times \frac{16R^2}{9} \times \left(2R - \frac{4R}{3}\right) = \frac{32\pi R^3}{81}$

EXERCISE 2.9

1. Divide 8 into two positive parts such that the sum of the square of one and the cube of the other is minimum.
2. Divide a into two parts such that the product of the pth power of one part and the qth power of the second part may be maximum.
3. Prove that the largest rectangle with a given perimeter is a square.
4. Given the perimeter of a rectangle, show that its diagonal is minimum when it is a square.
5. Prove that the perimeter of a right-angled triangle of given hypotenuse is a maximum when the triangle is isosceles.
6. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base
7. Find the point on the curve $y^2 = 4x$ which is nearest to the point $(2, -8)$.
8. Show that the surface area of a closed cuboid with square base and given volume is minimum when it is a cube.

Answers

1. 6, 2 **2.** $\frac{ap}{p+q}, \frac{aq}{p+q}$ **7.** $(4, -4)$

INTERESTING FACTS

- Maximum and minimum value that are found using Maxima and Minima are together known as Extrema (plural of extremum).
- A sculpture was displayed in the World Expo 2017 in Astana, Kazakhstan, which was named as Minima | Maxima, whose design was unique in itself.
- (https://architizer.com/projects/minima-maxima-1/)

VIDEO REFERENCES

Extreme values-I

APPLICATIONS TO REAL LIFE

- Applications in medicine while studying the effectiveness of drugs /spread of diseases (*i.e.*, after how much time the maximum efficiency has been observed).
- Decay study in nuclear energy sector.
- In business, industry uses this concept for maximising their profits or minimizing their loss, by estimating prices for items and also how many to keep in stock.
- Population Growth Curve.

SUBJECTIVE SOLVED QUESTIONS
(HOTS)

Example 1. *Let $f : [2, 5] \to R$ be continuous and differentiable on (2, 5). Assume that $f'(x) = (f(x))^2 + \pi$ for all $x \in (2, 5)$. Find $f(5) - f(2)$.*

Solution. Given that f is continuous on [2, 5] and differentiable on (2, 5).

Hence, by Lagrange's mean value theorem, there exists atleast one $c \in (2, 5)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c). \text{ Here } b = 5, a = 2$$

then $$\frac{f(5) - f(2)}{5 - 2} = (f(c))^2 + \pi$$

$$\frac{f(5) - f(2)}{3} \geq \pi \qquad [\because (f(c))^2 \geq 0 \ \forall \ c \in (2, 5)]$$

$$\therefore \quad f(5) - f(2) \geq 3\pi.$$

Example 2. *Prove that between any two roots of $e^x \cos x = 1$, there exists atleast one root of $e^x \sin x - 1 = 0$.*

Solution. Given that

$$e^x \cos x = 1$$

After rearranging, we have

$$\cos x = e^{-x}$$

Consider $$f(x) = \cos x - e^{-x}$$

i. cosine function and exponential function both are continuous and differentiable on $\mathbb{R}$, so on every interval say $[a, b]$.

ii. assuming that a, b are root of f.

Then $$f(a) = f(b) = 0$$

Hence by Rolle's theorem, $\exists\, c \in (a, b)$, such that $f'(c) = 0$

Now, $$f'(x) = -\sin x + e^{-x}$$

$$\Rightarrow \quad -\sin c + e^{-c} = 0$$

$$\Rightarrow \quad e^{-c} = \sin c$$

or $$1 = e^c \sin c$$

i.e., $$e^c \sin c - 1 = 0$$

This imply that $e^x \sin x = 1$ has one root $c \in (a, b)$ *i.e.*, between two roots of $e^x \cos x = 1$.

Example 3. *Prove that the quadratic equation $3px^2 + 2qx + r = 0$ has a root in $(0, 1)$ if $p + q + r = 0$.*

Solution. Let $$f(x) = px^3 + qx^2 + rx$$

i. Clearly $f(x)$ is continuous in $[0, 1]$, being a polynomial in x.

ii. $$f'(x) = 3px^2 + 2qx + r$$

$f(x)$ is differentiable in $(0, 1)$, again, being a polynomial in x.

iii. $$f(0) = p(0)^3 + q(0)^2 + r(0) = 0$$

$$f(1) = p(1)^3 + q(1)^3 + r(1) = p + q + r = 0$$

Hence, Rolle's theorem satisfied, there exists atleast one $c \in (0, 1)$ such that $f'(c) = 0$

$$f'(c) = 3pc^2 + 2qc + r$$

as $$f'(c) = 0 \quad \Rightarrow \quad 3pc^2 + 2qc + r = 0$$

which is quadratic in c and $c \in (0, 1)$.

Thus, $3px^2 + 2qx + r = 0$ has root in $(0, 1)$.

Example 4. *Let the function f be continuous on the closed interval $[a, b]$, differentiable on open interval (a, b) and $f'(x) = 0$ for all $x \in (a, b)$. Then show that f is constant on $[a, b]$.*

Solution. To show that f is constant on $[a, b]$, it is sufficient to prove that $f(x) = f(a)\ \forall\, x \in [a, b]$

Let $x \in [a, b]$ such that $x > a$.

Now, applying mean value theorem on $[a, x]$ as

i. f is continuous on $[a, b]$

$\therefore$ f is continuous on $[a, x]$.

ii. It is given that f is differentiable on (a, b)

$\therefore$ f is differentiable on (a, x).

By Lagrange's mean value theorem, there exists atleast one $c \in (a, x)$ such that

$$\frac{f(x)-f(a)}{x-a} = f'(c) \qquad ...(1)$$

Since, $\quad f'(c) = 0 \qquad [\because \; f'(x) = 0 \; \forall \; x \in (a, b)]$

From (1), $$\frac{f(x)-f(a)}{x-a} = 0$$

$$f(x) - f(a) = 0$$

$\Rightarrow \quad f(x) = f(a)$ for any $x \in [a, b]$.

$\therefore \quad f$ is constant on $[a, b]$.

Example 5. *Let* $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}$ *and let* $g(x) = x^2$ *for* $x \in [0, 1]$. *Then both f and g are differentiable on [0, 1] and* $g(x) > 0$ *for* $x \neq 0$. *Show that* $\lim_{x \to 0} f(x) = 0 = \lim_{x \to 0} g(x)$ *and that* $\lim_{x \to 0} \frac{f(x)}{g(x)}$ *does not exist.*

Solution. Here $$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0 \text{ for } x \neq 0$$

Also, $$g(x) = x^2 \; \forall \; x \in [0, 1]$$

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} x^2 = 0 \; \forall \; x$$

Now, for $x \neq 0$,

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x^2 \sin 1/x}{x^2} = \lim_{x \to 0} \sin \frac{1}{x}$$

$$= \sin(\infty)$$

$= $ oscillates between -1 and 1

$= $ Does not exist.

Example 6. *Using the remainder of Maclaurin polynomial of* n^{th} *order for f(x) defined as*

$$R_n(x) = \frac{x^{n+1}}{(n+1)!} f^{n+1}(c), \; n \ge 0, \; 0 \le c \le x.$$

What is the order of the Maclaurin polynomial at least required to get an absolute true error of atmost 10^{-6} *in the calculation of sin (0.1).*

Solution. Given that $$R_n(x) = \frac{x^{n+1}}{(n+1)!} f^{n+1}(c), \; n \ge 0, \; 0 \le c \le x$$

$$R_n(0.1) = \frac{(0.1)^{n+1}}{(n+1)!} f^{n+1}(c), \; n \ge 0, \; 0 \le c \le 0.1$$

Since derivative of $f(x)$ are simply $\sin x$ and $\cos x$ and $|\sin x| \le 1$ and $|\cos x| \le 1$

$\therefore \quad |f^{n+1}(c)| \le 1$

Now, $\quad R_n(0.1) \le \dfrac{(0.1)^{n+1}}{(n+1)!}(1)$

$$= \frac{(0.1)^{n+1}}{(n+1)!}$$

So, when $\quad R_n(0.1) < 10^{-6}$

i.e., $\quad \dfrac{(0.1)^{n+1}}{(n+1)!} < 10^{-6}$

This is possible only when $n \ge 4$.

But since Maclaurin polynomial for sin x only include odd terms, therefore, $n \ge 5$.

Example 7. *Let $f : (0, \infty) \to R$ be differentiable. Suppose that $\lim\limits_{x \to \infty} (f(x) + f'(x)) = L$. Show that $\lim\limits_{x \to \infty} f(x) = L$ and $\lim\limits_{x \to \infty} f'(x) = 0$.*

Solution. Try yourself.

Example 8. *Let $f : [a, b] \to R$ be thrice differentiable function such that $f(a) = f(b) = 0$ and $f'(a) = f'(b) = 0$, then prove that there exist $c \in (a, b)$ such that $f'''(c) = 0$.*

Solution. Here, $f : [a, b] \to R$ be thrice differentiable function.

$\Rightarrow$ f is continuous on $[a, b]$ and f is differentiable on (a, b) and also $f(a) = f(b) = 0$ then by Rolle's theorem, there exist atleast one $c_1 \in (a, b)$ such that

$$f'(c_1) = 0$$

Again applying Rolle's theorem for f'

i. f' is continuous on $[a, b]$ and, therefore, continuous on $[a, c_1]$ and $[c_1, b]$.

ii. f' is differentiable on (a, b), therefore, differentiable on (a, c_1) and (c_1, b)

iii. $f'(a) = f'(c_1) = f(b) = 0$

By Rolle's theorem, there exists atleast one $c_2 \in (a, c_1)$ and one $c_3 \in (c_1, b)$ such that

$$f''(c_2) = 0 \text{ for } c_2 \in (a, c_1)$$

and $\quad f''(c_3) = 0 \text{ for } c_3 \in (c_1, b)$

Similarly, applying Rolle's theorem on f'' on (c_2, c_3), f'' being continuous on $[c_2, c_3]$ and differentiable on (c_2, c_3) and $f''(c_2) = f''(c_3) = 0$ then there exists atleast one $c \in (c_2, c_3)$ such that

$$f'''(c) = 0 \text{ for } c \in (c_2, c_3)$$

Thus, $\quad f'''(c) = 0 \text{ for } c \in (a, b)$

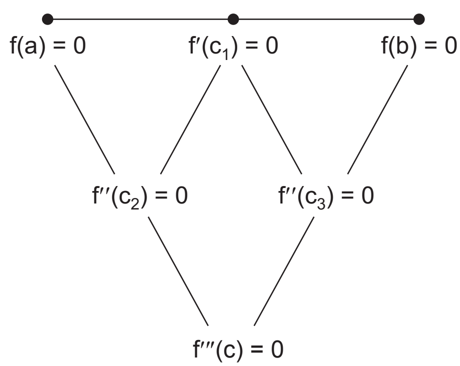

Fig. 2.16

Example 9. *A square piece of tin of side 24 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to form a box. What should be the side of the square to be cut off so that the volume of the box is maximum? Also find this maximum volume.*

Solution. Let each side of the square to be cut off = x

$\therefore$ For the box, length = $24 - 2x$

breadth = $24 - 2x$, height = x

Let V be the volume of the box

$$\therefore \quad V = x(24 - 2x)^2$$

$$\therefore \quad \frac{dV}{dx} = x \cdot 2(24 - 2x)(-2) + (24 - 2x)^2 \cdot 1 = (24 - 2x)(24 - 6x)$$

For maxima or minima, $\frac{dV}{dx} = 0$

$\Rightarrow$ $(24 - 2x)(24 - 6x) = 0$ $\Rightarrow$ $x = 4, 12$

$x = 4$ [$\because$ $x = 12$ cm is not possible]

Also, $$\frac{d^2V}{dx^2} = (24 - 2x)(-6) + (24 - 6x)(-2)$$

At $x = 4$ $$\frac{d^2V}{dx^2} = (24 - 8)(-6) = -\text{ve}$$

$\Rightarrow$ V is maximum when $x = 4$

$\therefore$ Volume is maximum when square of side 4 cm is cut from each corner.

$\therefore$ Maximum volume = $4(24 - 8)^2 = 1024$ cu. cm.

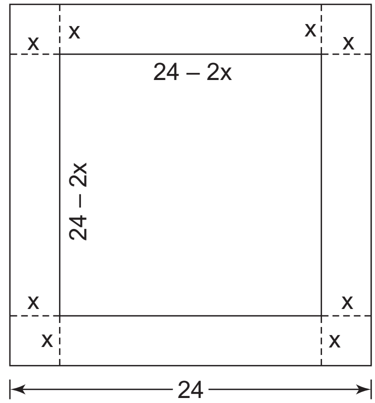

Fig. 2.17

Example 10. *A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m³. If building of tank costs ₹ 70 per sq. m, for the base and ₹ 45 per sq. m. for sides, what is the cost of least expensive tank?*

Solution. Let x mt and y mt be the sides of the base of the tank.

The depth of the tank is given to be 2 m

Volume of tank = $2xy$ m^3

$\Rightarrow$ $2xy = 8$

$\Rightarrow$ $xy = 4$ [$\because$ Volume of tank = 8 m^3 (given)]

Area of base of tank = $(x \times y)$ m^2

Now, it is given that cost of building the base of tank is ₹ 70 per sq. m.

$\therefore$ Cost of building the base of tank = ₹ $70\,xy$

Total area of the sides of the tank

$= (2x + 2x + 2y + 2y)$ m^2 = $4(x + y)$ m^2

It is given that the cost of building the sides of tank is ₹ 45 per sq. m.

$\therefore$ Cost of building the sides of the tank = ₹ $180(x + y)$

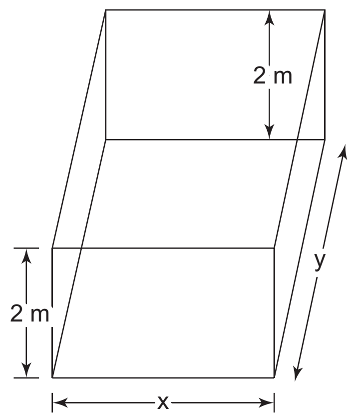

Fig. 2.18

Let C be the total cost of building the tank

$$\therefore \quad C = ₹\,[70xy + 180(x + y)]$$

$$= ₹\left[70(4) + 180\left(x + \frac{4}{x}\right)\right] \quad [\because \; xy = 4]$$

$$= ₹\left[280+180x+\frac{720}{x}\right]$$

$\therefore \qquad \dfrac{dC}{dx} = 180 - \dfrac{720}{x^2}$

For maxima or minima, $\dfrac{dC}{dx} = 0$

i.e., $\qquad 180 - \dfrac{720}{x^2} = 0 \Rightarrow x^2 = 4 \Rightarrow x = 2$ $\qquad$ [$\because$ x cannot be –ve]

Now, $\qquad \dfrac{d^2C}{dx^2} = \dfrac{1440}{x^3}$

At $x = 2$, $\qquad \dfrac{d^2C}{dx^2} = \dfrac{1440}{(2)^3} = +\text{ve}$

$\therefore$ C (*i.e.*, total cost) is minimum when $x = 2$

Hence, the cost of the least expensive tank = ₹ $\left[280+180(2)+\dfrac{720}{2}\right]$ = ₹ 1000.

SUMMARY

1. Necessary condition for applying the Rolle's theorem, mean value theorem, is that function should be continuous and differentiable.
2. Expansion of $f(x)$ in powers of x (Maclaurin's series)

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots$$

3. Expansion of function $f(x)$ about $x = a$ in powers of $(x - a)$ (Taylor's series)

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots$$

4. Indeterminate forms

i. $\dfrac{0}{0}$ $\qquad$ ii. $\dfrac{\infty}{\infty}$ $\qquad$ iii. $0 \times \infty$ $\qquad$ iv. $\infty - \infty$

v. 1^∞ $\qquad$ vi. 0^0 $\qquad$ vii. ∞^0

5. L'Hospital's Rule

If $\lim\limits_{x\to a} f(x) = f(a) = 0$ and $\lim\limits_{x\to a} g(x) = g(a) = 0$ then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a^+} \dfrac{f'(x)}{g'(x)}$

6. Standard Results on Limit

a. $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ $\qquad$ b. $\lim\limits_{x\to 0} \dfrac{\tan x}{x} = 1$ $\qquad$ c. $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x} = \log a$ $\qquad$ d. $\lim\limits_{x\to 0} \left(1+\dfrac{1}{x}\right)^x = e$

7. Maxima and Minima: For the function $f(x)$, we find $f'(x)$ and equate it to zero *i.e.*, $f'(x) = 0$ which gives the critical points of $f(x)$. Now on these critical points we check maxima and minima of $f(x)$.

Case I. First find $f''(x)$ on critical points,

If $f''(x) > 0$, then critical point is point of minima.

Case II. If $f''(x) < 0$, then critical point is point of maxima.

8. Some standard expansion of functions:

a. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

b. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

c. $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$

d. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

e. $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$, if $|x| < 1$

f. $\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}$

g. $(1 + x)^n$, for $|x| < 1$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots, |x| < 1$$

OBJECTIVE QUESTIONS

1. Find $\lim_{p \to \infty} \frac{p^5 \cdot p!}{5.6 \ldots (5 + p)}$

a. 4 ! b. 5 ! c. 0 d. ∞

2. Find $\lim_{x \to 0} \frac{\sin(x^2)}{x}$

a. ∞ b. -1 c. 0 d. 2^2

3. The value of $\lim_{x \to 0} \frac{\sqrt{\frac{1}{2}(1 - \cos 2x)}}{x}$

a. 1 b. -1 c. 0 d. does not exist

4. To verify Rolle's theorem which one is essential?

a. continuous and differentiable in open interval

b. continuous in open interval and differentiable in closed interval

c. continuous in closed interval and differentiable in open interval

d. continuous and differentiable in closed interval

5. When Rolle's theorem is verified for $f(x)$ on $[a, b]$, then there exist c such that

a. $c \in [a, b]$ such that $f'(c) = 0$

b. $c \in (a, b)$ such that $f'(c) = 0$

c. $c \in [a, b)$ such that $f'(c) = 0$

d. $c \in (a, b]$ such that $f'(c) = 0$

6. If the function $f(x) = x^2 - 8x + 12$ satisfies the condition of Rolle's theorem on (2, 6), find the value of c such that $f'(c) = 0$

a. 6 b. 4 c. 8 d. 2

7. If $f(x) = \dfrac{\sin x}{x}$, how many points exists such that $f'(c) = 0$ in the interval $[0, 18\pi]$

a. 8 b. 9 c. 17 d. 18

8. If $f(x) = \ln(10 - x^2)$, $x \in [-3, 3]$, find the point in the interval $[-3, 3]$ where slope of tangent is zero

a. 0

b. 2

c. Rolle's theorem is not applied, because function is not continuous in $[-3, 3]$

d. Rolle's theorem is not applied, because function is not differentiable in $[-3, 3]$

9. Find the point 'c' in the curve $f(x) = x^3 + x^2 + x + 1$ in the interval [0, 1], where slope of a tangent to a curve is equal to slope of line joining (0, 1) and (1, 4)

a. 0.64 b. 0.44 c. 0.54 d. 0.34

10. For the function $f(x) = x^3 + x + 1$, we do not have any Rolle's point. The coordinate axes are transformed by rotating them by 60 degree in anti-clockwise sense. The new Rolle's point is

a. $\sqrt{3}/2$

b. The function can never have a Rolle point

c. $\sqrt{3}$

d. $\sqrt{\dfrac{\sqrt{3}-1}{3}}$

11. Find the point on the curve $y = \log x$, tangent at which is parallel to the chord joining points (1, 0) and $(e, 1)$

a. e b. $e - 1$ c. $1 - e$ d. $-e$

12. If $f(a) = f(b)$ in mean value theorem, then it becomes

a. Leibnitz's theorem

b. Rolle's theorem

c. Taylor's series of a function

d. Cauchy's theorem

13. If $f(x) = \sin x \cos x$ is continuous and differentiable in the interval $(0, x)$, then

a. $1 < \dfrac{\cos x \,.\, \sin x}{x} < \sin 2x$

b. $1 < \dfrac{\cos x \,.\, \sin x}{x} < \cos 2x$

c. $1 < \dfrac{\cos x \,.\, \sin x}{x} < x \cos 2x$

d. $1 < \dfrac{\cos x \,.\, \sin x}{x} < 1 + \cos 2x$

14. On the interval [0, 1], the function $x^{25}(1 - x)^{75}$ takes its maximum value at

a. 0

b. 1/2

c. 1

d. 1/4

15. Find the co-efficient of x^2 in the Taylor's series about $x = 0$ for $f(x) = e^{-x^2}$

a. 1/4

b. – 1

c. 1/2

d. –2

16. $f(x) = 1 + 2x^2 + 4x^4 + 6x^6 + \ldots + 100x^{100}$ is polynomial in a real variable x, then $f(x)$ has

a. neither maximum nor minimum

b. only one maximum

c. only one minimum

d. one maximum and one minimum

17. Expansion of function $f(x)$ using Maclaurin's series is

a. $f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots$
b. $1+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots$
c. $f(0)-\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\ldots$
d. $f(1)+\frac{x}{1!}f'(1)+\frac{x^2}{2!}f''(1)+\frac{x^3}{3!}f'''(1)+\ldots$

18. The real number x when added to its inverse gives the minimum value of the sum at x equal

a. –2 b. 2 c. 1 d. –1

19. If the function $f(x) = 2x^2 - 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its maximum and minimum at p and q respectively such that $p^2 = q$, then 'a' is equal to

a. 1/2 b. 3 c. 1 d. 2

20. The necessary condition for the Maclaurin's expansion to be true for the function $f(x)$ is

a. it should be continuous
b. it should be differentiable
c. it should exists at every point
d. it should be continuous and differentiable both

Answers

1. a	**2.** c	**3.** d	**4.** c
5. b	**6.** b	**7.** d	**8.** a
9. c	**10.** d	**11.** b	**12.** b
13. b	**14.** d	**15.** b	**16.** c
17. a	**18.** c	**19.** c	**20.** d

SUBJECTIVE UNSOLVED QUESTIONS
(HOTS)

1. Let $f(x)$ be a continuous and differentiable function on R, then show that between any two real root of $f(x)$, there exists atleast one root of $f'(x)$.

2. Let $f: [a, b] \to R$ be thrice differentiable function such that $f(a) = f(b) = f'(a) = f''(a) = 0$, then prove that there exists $c \in (a, b)$ such that $f'''(c) = 0$.

3. Using mean value theorem, prove that $|\sin x - \sin y| \le |x - y|\ \forall\ x, y \in R$.

4. Let $f: (0, \infty) \to R$ be a differentiable function then prove that, for any $a > 0$, if

$$\lim_{x\to\infty}(af(x) + f'(x)) = L, \text{ then } \lim_{x\to\infty} f(x) = \frac{L}{a}.$$

5. If $a_0 + a_1 + \ldots + a_n = 0$ where $a_i \in R$, $1 \le i \le n$ then show that $a_0 + 2a_1x + \ldots + (n+1)a_n x^n = 0$ has atleast one real root in (0, 1).

6. Using mean value theorem, prove the inequality $\frac{y-x}{y} < \log\left(\frac{y}{x}\right) < \frac{y-x}{x}$ for all $0 < x < y$.

7. Let $f(x) \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \ne 0 \\ 0, & x = 0 \end{cases}$ and let $g(x) = \sin x\ \forall\ x \in R$. Show that $\lim_{x\to 0}\frac{f(x)}{g(x)} = 0$ but $\lim_{x\to 0}\frac{f'(x)}{g'(x)}$ does not exist.

8. Let $f : R \to R$ be a differentiable function such that $\left|\frac{df(x)}{dx}\right| \le 4$ and $f(0) = 0$ then prove that $f(1) \in [-4, 4]$ and $f(2) \in [-8, 8]$.
9. Let I be an interval and $f : I \to R$ be differentiable on I. Show that if the derivative f' is never 0 on I, then either $f'(x) > 0 \ \forall\ x \in I$ or $f'(x) < 0 \ \forall\ x \in I$.
10. Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle 30° is $\frac{4}{81}\pi h^3$.
11. A water tank has the shape of an inverted right circular cone with its axis vertical and vertex lowermost. Its semi-vertical angle is $\tan^{-1}(0.5)$. Water is poured into it at a constant rate of 5 cubic metre per hour. Find the rate at which the level of the water is rising at the instant when the depth of water in the tank is 4 m.

PRACTICAL

1. Sketch the graph of sine and cosine functions in $[0, 2\pi]$ using MATLAB tool.
2. Plot a graph for e^{3x} on R using MATLAB tool.
3. In M.S. Excel, draw $[x]$, greatest integer function in the interval $[0, 5]$.

ACTIVITY

1. Write a MATLAB code to verify Rolle's theorem for the function
   i. x^2 in the interval $[-3, 3]$
   ii. $(x + 2)^3 * (x - 3)^4$ in the interval $[-2, 3]$.

   Also Plot the curve for the same.

KNOW MORE

1. The value of $\lim_{x \to 0} \left\{\frac{a^x + b^x + c^x}{3}\right\}^{1/x}$ is

   a. abc b. $(abc)^{1/3}$ c. $(abc)^{1/8}$ d. $\frac{1}{abc}$

2. The product of minimum value of x^x and maximum value of $\left(\frac{1}{x}\right)^x$ is

   a. e b. $\frac{1}{e}$ c. 1 d. e^2

3. The expansion of $e^{\sin x}$ is ......

   a. $1 + x + \frac{x^2}{2} + \frac{x^4}{8} + \ldots$ b. $1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \ldots$

   c. $1 + x - \frac{x^2}{2} + \frac{x^4}{8} + \ldots$ d. $1 + x + \frac{x^2}{2} - \frac{x^5}{10} + \ldots$

4. Find relation between 'a' and 'b' such that the following limit obtain after a single application of L'Hospital Rule on $\underset{x \to 0}{\text{lt.}} \dfrac{ae^x + be^{2x}}{be^x + ae^{2x}}$.

a. $b/a = 2$ b. $a/b = 2$ c. $a = b$ d. $a = -b$

5. Find how many rounds of differentiation are required to have finite limit for

$\underset{x \to 0}{\text{lt.}} \dfrac{\cos(ax) + \cos(bx) - 2\cos(cx)}{\cos(ax) + 2\cos(bx) - 3\cos(cx)}$, given that $a \neq b \neq c$

a. 3 b. 0 c. 2 d. 4

Answers

1. b	**2.** c	**3.** b	**4.** d
5. c			

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3 Matrices

UNIT SPECIFICS

This unit discusses the topics – matrices and their types ,operations on matrices, properties of matrices, vectors, operation on vectors, determinant and its properties and applications, linear systems of equations, rank of a matrix, inverse of a matrix, Cramer's rule, Gauss elimination and Gauss-Jordan elimination in length. All the concepts have been explained so as to develop an interdisciplinary approach in students.

RATIONALE

Matrices are most widely used in the solution of system of linear algebraic equations, linear differential equations and non-linear differential equations. Many physical operations such as magnification, rotation and reflection can also be represented mathematically with the help of matrices. In encryption, we use it to scramble data for security purpose to encode and to decode this data. Matrices are used in representing the real world data like traits of people, population, habits etc. In robotics and automation, matrices are the base elements for the robot movements. Linear system of equations are used in many areas such as age-problem, speed-time, etc.

PRE-REQUISITES

1. Basic knowledge of Algebra of Matrices.
2. Familiar with the concept of adjoint, inverse and determinant with their properties.
3. Know the idea of formation of matrices from the linear system of equations.

UNIT OUTCOMES

After completion of this unit, students will be able to:

U3-01: Recognize consistent and inconsistent systems of linear equations and compute their solutions by the row echelon form of the augmented matrix, using rank.

U3-02: Familiarize themselves with the concepts of algebraic and geometric representation of vector in R^n and their operations, including addition, scalar multiplication and dot product.

U3-03: Recognize and use equivalent statements regarding invertible matrices, pivot positions and solution of homogeneous and non-homogeneous system of linear equations.

U3-04: Apply Gauss Elimination method, Gauss Jordan method and Cramer's Rule to compute the existence and uniqueness of solutions for system of linear equations.

MAPPING OF UNIT OUTCOMES WITH COURSE OUTCOMES

Unit 3 Outcomes	EXPECTED MAPPING WITH COURSE OUTCOMES (1- Weak Correlation; 2- Medium Correlation; 3- Strong Correlation)				
	CO-1	CO-2	CO-3	CO-4	CO-5
U3-01	–	–	3	1	1
U3-02	–	–	1	3	2
U3-03	–	–	3	2	1
U3-04	–	–	3	–	–

HISTORY

Historically, the study of matrices and determinants goes back to the second century BC although traces can be seen back to the fourth century BC. It is not surprising that the beginning of matrices and determinants should arise through the study of systems of linear equations. The Babylonians studied problems which lead to simultaneous linear equations and some of these are preserved in clay tablets which survived.

However it was not until near the end of the 17th Century that the ideas reappeared and development really got underway, it was not the matrix but a certain number associated with a square array of numbers called the determinant that was first recognized. Only gradually did the idea of the matrix as an algebraic entity emerge. The term matrix was introduced by the 19th-century English mathematician James Sylvester, but it was his friend the mathematician Arthur Cayley who developed the algebraic aspect of matrices in two papers in the 1850s. Cayley first applied them to the study of systems of linear equations, where they are still very useful. They are also important because, as Cayley recognized, certain sets of matrices form algebraic systems in which many of the ordinary laws of arithmetic (*e.g.*, the associative and distributive laws) are valid but in which other laws (*e.g.*, the commutative law) are not valid. Matrices have also come to have important applications in computer graphics, where they have been used to represent rotations and other transformations of images.

An equation means nothing to me unless it expresses a thought of God.

—Srinivasa Ramanujan (1887-1920)

INTRODUCTION

The word matrices is plural of the word matrix. Arthur Cayley, first person to introduce the concept of matrices in 1860.

The study of matrices was originated from the idea of various types of linear problems. It has a special relationship with the system of linear equations which occur in many engineering processes. It provides an important list in the study and development of linear algebra. The matrices also occur in presentation of linear system models in applied engineering and control systems. Matrices are widely used in the study of every branch of Mathematics, Science and Engineering.

3.1 DEFINITION

A set of mn numbers (real or complex) arranged in the form of rectangular array having m rows (horizontal lines) and n columns (vertical lines) is called $m \times n$ matrix (read as 'm by n matrix' or 'matrix of order m by n' or 'matrix of type m by n'.

A matrix is generally represented by the symbol $[a_{ij}]$ or (a_{ij}) or $\| a_{ij} \|$.

A matrix is usually denoted by a single capital letter A, B, C etc.

Thus, an $m \times n$ matrix 'A' may be written as

$$A = [a_{ij}]_{m\times n} \quad \text{or} \quad A = [a_{ij}]$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{where } i = 1, 2, ..., m \text{ and } j = 1, 2, ..., n$$

Each $m \times n$ matrix has $m.n$ number of elements.

> **Note:** Each entry in the matrix is called an element of the matrix.

For example: Let us consider a set of simultaneous system of equations

$$2x + 3y + 3z + 2t = 0$$
$$3x + 2y + 5z + 3t = 0$$
$$4x + 5y + 6z + 7t = 0$$
$$2x + 3y + 4z + 5t = 0$$

Now, we write the coefficients of x, y, z and t of the above equations and enclose them with in the brackets and then, we get

$$A = \begin{bmatrix} 2 & 3 & 3 & 2 \\ 3 & 2 & 5 & 3 \\ 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 \end{bmatrix}$$

The above system of numbers arranged in a rectangular array of rows and columns and bounded by the brackets, is called a matrix.

It has got 4 rows and 4 columns and in all $4 \times 4 = 16$ elements. It is termed as 4×4 matrix.

In the double subscripts of an element, the first subscript determines the row and the second subscript determines the column in which the element lies, like a_{ij} lies in i^{th} row and j^{th} column.

3.1.1 Various Types of Matrices

1. **Real Matrix:** A matrix is said to be real matrix if all its elements are real.

e.g. $\begin{bmatrix} 1 & 0 & \sqrt{3} \\ 1/2 & 2 & 1 \end{bmatrix}_{2\times 3}$ is a real matrix.

2. **Complex Matrix:** A matrix whose elements may contain complex number is called a complex matrix.

   e.g. $\begin{bmatrix} 1 & i \\ -i & 0 \end{bmatrix}_{2\times 2}$ is a complex matrix.

3. **Row Matrix:** A matrix which has only one row and any number of columns, is called a row matrix.

   e.g. $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}_{1\times 4}$ is a row matrix.

4. **Column Matrix:** A matrix which has only one column and any number of rows is called a column matrix.

   e.g. $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}_{4\times 1}$ is a column matrix.

5. **Null Matrix or Zero Matrix:** A matrix which has all its elements zero is called a null matrix or zero matrix.

   e.g. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$ are null matrices.

6. **Square Matrix:** A matrix of order $m \times n$ is said to be a square matrix if $m = n$ *i.e.* the number of rows is equal to the number of columns.

   e.g. $\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}_{2\times 2}$ is a square matrix.

7. **Rectangular Matrix:** A matrix of order $m \times n$ is said to be a rectangular matrix if $m \neq n$ *i.e.* the number of rows is not equal to the number of columns.

   e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix}_{2\times 3}$ is a rectangular matrix.

8. **Diagonal Matrix:** A square matrix is called a diagonal matrix if all its non-diagonal elements are zero.

   Suppose $A = [a_{ij}]_{n\times n}$ and if $a_{ij} = 0$ for $i \neq j$, then 'A' is diagonal matrix of order $n \times n$.

   Diagonal matrix also written as Diag $[a_{11}, \quad a_{22}, \quad \ldots, \quad a_{nn}]$

   e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times 3}$ is a diagonal matrix.

9. **Unit Matrix or Identity Matrix:** A square matrix is said to be a unit matrix if all its diagonal elements are unity and non-diagonal elements are zero.

e.g. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$ are unit matrices.

Unit matrix of order n is also denoted as $I_{n\times n}$.

10. **Singular and Non-singular Matrices:** A square matrix 'A' is called a singular matrix if $|A| = 0$ *i.e.* if determinant formed by the elements of 'A' is zero.

    e.g. For $A = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$, we have $|A| = 0$ (singular matrix)

    If $|A| \neq 0$, (determinant not equal to zero) then the matrix 'A' is called as non-singular matrix.

    e.g. For $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$, we have $|A| = -10 \neq 0$ (non-singular matrix)

11. **Symmetric Matrix:** A square matrix $A = [a_{ij}]$ is called a symmetric matrix if

    $$a_{ij} = a_{ji} \; \forall \; i \text{ and } j$$

    or $A = A'$ (A' = Transpose of A)

    e.g. $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & e \end{bmatrix}_{3\times3}$, $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}_{3\times3}$ are symmetric matrices.

12. **Skew-symmetric Matrix:** A square matrix $A = [a_{ij}]$ is said to be a skew-symmetric matrix if

    a. $a_{ij} = -a_{ji} \quad \forall \; i$ and j or $A = -A'$

    b. All diagonal elements are zero

    e.g. $\begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}_{3\times3}$ is a skew-symmetric matrix.

13. **Transpose of a Matrix:** Let $A = [a_{ij}]_{m\times n}$. Then the $n \times m$ matrix obtained from A by changing its rows into columns and its columns into rows is called the transpose of A and is denoted by A' or A^T.

    Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 6 & 5 \end{bmatrix}$, then $A^T = A' = \begin{bmatrix} 1 & 2 & 6 \\ 3 & 4 & 5 \end{bmatrix}$

    If $B = [1 \quad 2 \quad 3]$, then $B' = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

    Properties of Transpose of a Matrix: If A' and B' be transposes of A and B respectively then,

    a. $(A')' = A$

    b. $(A + B)' = A' + B'$

    c. $(kA)' = kA'$, k being any number

d. $(AB)' = B'A'$

e. $(ABC)' = C'B'A'$

14. **Orthogonal Matrix:** A square matrix 'A' is called an orthogonal matrix if the product of the matrix 'A' and its transpose A' becomes an identity matrix *i.e.* $A \,.\, A' = I$ where I is an identity matrix.

e.g. For $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$, then $A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$, we have

$A \,.\, A' = I$ [Students can verify it at their parts]

then 'A' is an orthogonal matrix.

15. **Triangular Matrix:** A square matrix is said to be a triangular matrix if all the elements lying above or below the leading principal diagonal are zero.

There are two types of triangular matrix.

a. **Upper triangular matrix:** A square matrix all of whose elements below the leading diagonal are zero, is called an upper triangular matrix.

e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ ← leading diagonal

is an upper triangular matrix.

b. **Lower triangular matrix:** A square matrix all of whose elements above the leading diagonal are zero, is called a lower triangular matrix.

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 7 \end{bmatrix}$ ← leading diagonal

is a lower triangular matrix.

16. **Conjugate of a Matrix:**

Let $A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$

Conjugate of matrix A is denoted by $\overline{A}$.

$\therefore$ $\overline{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & i & 3+2i \end{bmatrix}$

Remark: Transpose of the conjugate of a matrix A is denoted by A^{θ}.

Let $A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$

$\overline{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & i & 3+2i \end{bmatrix}$

$$(\bar{A})' = \begin{bmatrix} 1-i & 7-2i \\ 2+3i & i \\ 4 & 3+2i \end{bmatrix}$$

or $$A^{\theta} = \begin{bmatrix} 1-i & 7-2i \\ 2+3i & i \\ 4 & 3+2i \end{bmatrix}$$

17. **Unitary Matrix:** A square matrix A is said to be unitary if $A^{\theta}A = 1$

e.g. $$A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}, \quad A^{\theta} = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}$$

$$A^{\theta}A = I.$$

18. **Hermitian Matrix:** A square matrix $A = [a_{ij}]$ is called Hermitian matrix, if every $(ij)^{\text{th}}$ element of 'A' is equal to the conjugate complex $(ji)^{\text{th}}$ element of A.

In other words, $a_{ij} = \bar{a}_{ji}$

e.g. $$A = \begin{bmatrix} 1 & 2+3i & 3+i \\ 2-3i & 2 & 1-2i \\ 3-i & 1+2i & 5 \end{bmatrix}$$

Necessary and sufficient condition for a matrix A to be Hermitian is that $A = A^{\theta}$ *i.e.*, conjugate transpose of A.

or $$A = (\bar{A})'.$$

19. **Skew-Hermitian Matrix:** A square matrix $A = [a_{ij}]$ is called a skew-Hermitian matrix if every $(ij)^{\text{th}}$ element of A is equal to negative conjugate complex of $(ji)^{\text{th}}$ element of A.

In other words, $a_{ij} = -\bar{a}_{ji}$

All the elements in the principal diagonal will be of the form.

$$a_{ii} = -\bar{a}_{ii} \quad \text{or} \quad a_{ii} + \bar{a}_{ii} = 0$$

if $a_{ii} = a + ib$ then $\bar{a}_{ii} = a - ib$

$$(a + ib) + (a - ib) = 0 \quad \text{or} \quad 2a = 0 \quad \text{or} \quad a = 0$$

so a_{ii} is purely imaginary or $a_{ii} = 0$

Hence all the diagonal elements of a skew-Hermitian matrix are either zero or purely imaginary.

e.g. $$\begin{bmatrix} i & 2-3i & 4+5i \\ -(2+3i) & 0 & 2i \\ -(4-5i) & 2i & -3i \end{bmatrix}$$

The necessary and sufficient condition for a matrix A to be skew-Hermitian is that,

$$A^{\theta} = -A$$

$$(\bar{A})' = -A$$

20. **Idempotent Matrix:** A matrix, such that $A^2 = A$ is called idempotent matrix.

e.g. If $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$, then

$$A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= A$$

21. **Periodic Matrix:** A matrix A will be called a periodic matrix, if

$$A^{k+1} = A$$

where 'k' is a +ve integer. If k is the least +ve integer, for which $A^{k+1} = A$, then k is said to be period of A. If we choose $k = 1$, we get $A^2 = A$ and we call it to be an idempotent matrix.

22. **Nilpotent Matrix:** A matrix will be called a nilpotent matrix, if $A^k = 0$ (null matrix), where k is a +ve integer; if however k is the least +ve integer for which $A^k = 0$, then k is the index of the nilpotent matrix.

e.g. $$A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}, \quad A^2 = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

'A' is a nilpotent matrix whose index is 2.

23. **Involutory Matrix:** A matrix 'A' will be called an involutory matrix if $A^2 = I$, unit matrix.

Since $I^2 = I$ (always)

$\therefore$ unit matrix is involutory.

24. **Trace of a Matrix:** Let A be a square matrix of order n. The sum of the elements lying along principal diagonal is called the trace of A denoted by Tr (A).

Thus if $A = [a_{ij}]_{n\times n}$, then

$$\text{Tr}(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \dots a_{nn}$$

Let $$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 6 & 5 \end{bmatrix}$$

Then, trace (A) = tr $(A) = 1 + (-3) + 5 = 3$

Properties of Trace of a Matrix: Let A and B be two square matrices of order n and λ be a scalar. Then,

a. tr $(\lambda A) = \lambda$ tr A

b. tr $(A + B)$ = tr A + tr B

c. tr (AB) = tr (BA)

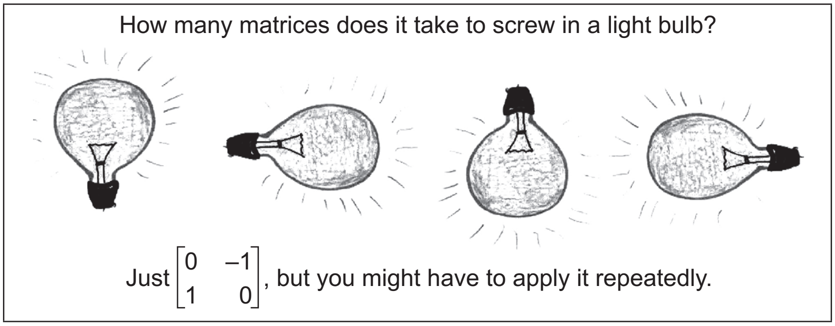


Fig. 3.1

3.1.2 Operation on Matrices

3.1.2.1 Addition of Matrices

The operation 'Addition' on two matrices is performed if they are of same order. Suppose '*A*' and '*B*' are two matrices of same order, then the sum of these two matrices is obtained by adding the corresponding elements of '*A*' and '*B*'.

It is denoted as $A + B$

e.g. If $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}_{2\times 3}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times 3}$, then

$$A + B = \begin{bmatrix} 2+1 & 3+2 & 1+1 \\ 1+1 & 2+1 & 3+1 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 2 \\ 2 & 3 & 4 \end{bmatrix}_{2\times 3}$$

In general, if $A = [a_{ij}]$ and $B = [b_{ij}]$, then

$$A + B = [a_{ij} + b_{ij}].$$

3.1.2.2 Properties of Matrix Addition

Matrices of the same order only can be added or subtracted.

a. **Commutative law:** Two matrices of same order can be added in any order *i.e.* commutative law holds in matrix addition. If '*A*' and '*B*' are two matrices, then

$$A + B = B + A \qquad \text{(holds)}$$

(Students can verify it by taking two matrices of same order)

b. **Associative law:** If we have three matrices '*A*', '*B*' and '*C*' of same order, then associativity property holds under addition, *i.e.*,

$$A + (B + C) = (A + B) + C \qquad \text{(students can verify it)}$$

3.1.2.3 Subtraction of Matrices

The operation 'subtraction' on two matrices is performed, if they are of same order.

Suppose '*A*' and '*B*' are two matrices of same order, then the difference of two matrices is obtained by subtracting each element of the second matrix from the corresponding elements of the first matrix.

It is denoted by $A - B$

e.g. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix}_{3\times 2}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}_{3\times 2}$, then

$$A - B = \begin{bmatrix} 1-1 & 1-1 \\ 1-2 & 2-2 \\ 2-3 & 3-1 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ -1 & 2 \end{bmatrix}_{3\times 2}$$

is the difference of two matrices 'A' and 'B'.

3.1.2.4 Scalar Multiplication of a Matrix

If a matrix is multiplied by a scalar quantity k, then each element of it is multiplied by k.

e.g. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}_{3\times 3}$, then

$$2A = 2\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}_{3\times 3} = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \\ 2 & 4 & 4 \end{bmatrix}_{3\times 3}$$

3.1.2.5 Multiplication of Matrices

Product of two matrices 'A' and 'B' is possible only if the number of columns in 'A' is equal to the number of rows in 'B'.

Let $A = [a_{ij}]_{p\times q}$ and $B = [b_{jk}]_{q\times r}$, then the product AB is defined as

$$C = [c_{ik}]_{p\times r}$$

where $$c_{ik} = a_{i1}\, b_{1k} + a_{i2}\, b_{2k} + \ldots\ldots + a_{in}\, b_{nk}$$

$$c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$$

and we can write, $$C = AB$$

e.g. a. If we have $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}_{2\times 2}$ and $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix}_{2\times 3}$, then

$$AB = \begin{matrix} R_1 \\ R_2 \end{matrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\overset{\begin{matrix} c_1 & c_2 & c_3 \end{matrix}}{\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix}}$$

$$= \begin{bmatrix} R_1c_1 & R_1c_2 & R_1c_3 \\ R_2c_1 & R_2c_2 & R_2c_3 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} 5 & 3 & 6 \\ 2 & 1 & 2 \end{bmatrix}_{2\times 3}$$

3.1.2.6 Properties of Matrix Multiplication

a. Multiplication of matrices is not commutative *i.e.* If 'A' and 'B' are two matrices, then

$$AB \neq BA \qquad \text{(need not to be equal)}$$

b. Matrix multiplication is associative, if confirmability is assured, *i.e.* for three matrices 'A', 'B' and 'C', we have

$$A(BC) = (AB)C$$

c. Matrix multiplication is distributive with respect to addition. For three matrices A, B and C, we have

$$A(B + C) = AB + AC$$

d. Multiplication of a matrix 'A' by a unit matrix 'I' is a matrix 'A' itself. *i.e.*

$$AI = IA = A$$

e. Multiplicative inverse of a matrix 'A' exists if $|A| \neq 0$ *i.e.*

$$AA^{-1} = A^{-1}A = I$$

For example: If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}_{3\times3}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}_{3\times2}$

then obtain the product AB and explain why BA is not defined?

Solution. Since the number of columns of A (3×3) = number of rows in B (3×2)

therefore the product AB is defined

$$AB = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}_{3\times3} \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}_{3\times2} = \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}_{3\times2}$$

Now, for BA, the number of columns of B $(3 \times 2) \neq$ the number of rows of A (3×3) therefore, the product BA is not defined.

> **Note:** 1. If A and B are two matrices of order $m \times n$ and $p \times q$ respectively, then the product AB is possible only when $n = p$ and the order of AB is $m \times q$.
>
> 2. For two matrices 'A' and 'B' if the product AB exist, then the product BA may or may not exist.

Pictorial Representation

1. Step by step visualization of matrix multiplication: http://matrixmultiplication.xyz

EXERCISE 3.1

1. Given $A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then find

a. $2A + 3B$ b. $3A - 4B$

2. Two matrices A and B are such that $3A - 2B = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$ and $-4A + B = \begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix}$, then find A and B.

3. If $A = \text{diag.}\ [2, \quad 9, \quad 4]$ and $B = \text{diag.}\ [-3, \quad 7, \quad 6]$, then find

a. $A + B$ b. $A - B$ c. $7A + 2B$ d. $9A - 11B$

4. Given $A = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$, then find the matrix X such that $2A + 3X = 5B$.

5. Given $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$, then find

a. AB b. BA

Also show that $AB \neq BA$.

Answers

1. a. $\begin{bmatrix} 3 & 10 & 3 \\ 8 & 3 & 6 \\ 2 & 2 & 13 \end{bmatrix}$ b. $\begin{bmatrix} -4 & -2 & -4 \\ -5 & -4 & 9 \\ 3 & 3 & -6 \end{bmatrix}$ **2.** $A = \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & -2 \\ 4 & -1 \end{bmatrix}$

3. a. diag. $[-1, \quad 16, \quad 10]$ b. diag. $[5, \quad 2, \quad -2]$

c. diag. $[8, \quad 77, \quad 40]$ d. diag. $[51, \quad 4, \quad -30]$

4. $X = \begin{bmatrix} 12 & 4/3 \\ 4 & -14/3 \\ 25/3 & 28/3 \end{bmatrix}$ 5. a. $\begin{bmatrix} 3 & 12 & 11 \\ 4 & 13 & 8 \\ 0 & -1 & 5 \end{bmatrix}$ b. $\begin{bmatrix} 11 & 9 & 13 \\ 3 & 2 & 4 \\ 0 & 5 & 8 \end{bmatrix}$

3.2 VECTORS

Any ordered n-tuple of numbers is called n-vector. By an ordered n-tuple, we mean a dot consisting of n numbers in which the place of each number is fixed. If $a_1, a_2, a_3, ..., a_n$ are any n-numbers, then the ordered n-tuple defined as $X = (a_1, a_2, a_3, ..., a_n)$ is called n-vector.

The numbers a_i are called the coordinates, components, entries or elements of X. A vector may be written either as a row vector or as a column vector.

Let $X = (a_1, a_2, ..., a_n)$ and $Y = (b_1, b_2, ..., b_n)$ be two vectors, then they are equal *i.e.* $X = Y$ if and only if their corresponding components are same, *i.e.*, $a_i = b_i$ for $i = 1, 2, ..., n$.

If 'A' be a matrix of order $m \times n$, then each row of 'A' will be of order n-tuple vector and each column of A will be of order m-tuple vector.

e.g. a. $(1, \quad 2, \quad 5)$ is a row vector

b. $\begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}$ is a column vector.

3.2.1 Operations on Vectors

3.2.1.1 Addition of Vectors

Let $X = (a_1, a_2, ..., a_n)$ and $Y = (b_1, b_2, ..., b_n)$ be the set of two vectors, then addition of two vectors, X and Y will be $X + Y = (a_1 + b_1, a_2 + b_2, ..., a_n + b_n)$

e.g. Let $X = (1, 2, 3)$ and $Y = (1, 1, 1)$, then $X + Y = (2, 3, 4)$.

3.2.1.2 Subtraction of Vectors

Let $X = (a_1, a_2, ..., a_n)$ and $Y = (b_1, b_2, ..., b_n)$ be the two vectors, then the difference of two vectors X and Y is denoted as $X - Y = (a_1 - b_1, a_2 - b_2, ..., a_n - b_n)$

e.g. Let $X = (2, 3, 5)$ and $Y = (1, 2, 6)$, then $X - Y = (1, 1, -1)$.

3.2.1.3 Scalar Multiplication of a Vector

Let $X = (a_1, a_2, ..., a_n)$ be any n-tuple vector and k is any scalar quantity, then scalar multiplication of a vector X with k is defined as

$$kX = k(a_1, a_2, ..., a_n)$$
$$= (ka_1, ka_2, ..., ka_n)$$

e.g. Let $X = (1, 2, 3)$, then

$$6X = 6(1, 2, 3) = (6, 12, 18).$$

3.2.1.4 Dot (Inner) Product

Let $X = (a_1, a_2, ..., a_n)$ and $Y = (b_1, b_2, ..., b_n)$ are two vectors, then the dot product of X and Y will be defined as

$$X \,.\, Y = (a_1, a_2, ..., a_n)\,(b_1, b_2, ..., b_n)$$
$$= (a_1b_1 + a_2b_2 + ... + a_nb_n)$$

e.g. Let $X = (2, 5, 4)$ and $Y = (1, 2, 5)$, then

$$XY = (2, 5, 4) \,.\, (1, 2, 5)$$
$$= (2)\,(1) + (5)\,(2) + (4)\,(5) = 2 + 10 + 20 = 32$$

SOME SOLVED EXAMPLES

Example 3.1. *Given $X = (1, 2, 5, 4)$ and $Y = (0, 3, 9, 12)$, then find $2X - Y$.*

Solution. Given $X = (1, 2, 5, 4)$ and $Y = (0, 3, 9, 12)$, then

$$2X - Y = 2(1, 2, 5, 4) - (0, 3, 9, 12)$$
$$= (2, 4, 10, 8) - (0, 3, 9, 12)$$
$$= (2, 1, 1, -4)$$ **Answer**

Example 3.2. *Given $X = (1, 5, 2)$ and $Y = (1, 10, 11)$, then find*

a. $X + Y$ b. $9X$ c. $2X - 5Y$

Solution. We have $X = (1, 5, 2)$ and $Y = (1, 10, 11)$, then

a. $X + Y = (1, 5, 2) + (1, 10, 11) = (2, 15, 13)$

b. $9X = 9(1, 5, 2) = (9, 45, 18)$

c. $2X - 5Y = 2(1, 5, 2) - 5(1, 10, 11)$
$= (2, 10, 4) - (5, 50, 55) = (-3, -40, -51).$

Example 3.3. *If $X = (9, 4, 5, 10)$ and $Y = (0, -3, 2, -1)$, then find*

a. $X . X$ b. $Y . X$

Solution. Given $X = (9, 4, 5, 10)$ and $Y = (0, -3, 2, -1)$

a. $X . X = (9, 4, 5, 10) . (9, 4, 5, 10)$
$= (9)(9) + (4)(4) + (5)(5) + (10)(10)$
$= 81 + 16 + 25 + 100 = 222$

b. $Y . X = (0, -3, 2, -1) . (9, 4, 5, 10) = -12$ (Try yourself)

EXERCISE 3.2

1. Find 'a' and 'b', where

i. $(a, 3) = (2, a + b)$ ii. $(4, b) = a(2, 3)$

2. If $X = (2, 3, 0, 5)$, $Y = (0, 6, -1, 9)$ and $Z = (1, 1, 1, 0)$, then find

i. $X + Y$ ii. $Y - 3Z$

3. Determine which of the following vectors are equal?

$$X_1 = (1, 2, 3), X_2 = (1, 3, 2), X_3 = (2, 3, 1), X_4 = (2, 3, 1).$$

4. If $X = \begin{bmatrix} 5 \\ 3 \\ -4 \end{bmatrix}$, $Y = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$ and $Z = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$, then find

i. $5X - 2Y$ ii. $-2X + 4Y - 3Z.$

5. If $X = (-2, 5, \sqrt{10}, 3, 4)$ and $Y = (1, -2, \sqrt{10}, -4, 3)$, then find

i. $X . X$ ii. $Y . Y$

6. If two vectors $(a + b, a - b) = (5, 3)$ is given, then find the value of 'a' and 'b'.

Answers

1. i. $a = 2, b = 1$ ii. $a = 2, b = 6$ **2.** i. $(2, 9, -1, 14)$ ii. $(-3, 3, -4, 9)$

3. $X_3 = X_4$ **4.** i. $\begin{bmatrix} 27 \\ 5 \\ -24 \end{bmatrix}$ ii. $\begin{bmatrix} -23 \\ 17 \\ 22 \end{bmatrix}$

5. i. 64 ii. 40 **6.** $a = 4, b = 1$

3.3 ELEMENTARY OPERATIONS (TRANSFORMATION)

Any one of the following operations on a matrix is called an elementary transformation or E-transformation.

1. Interchange of any two rows (or columns). The interchange of i^{th} and j^{th} rows is denoted by R_{ij} or $R_i \leftrightarrow R_j$.
   Similarly, the interchange of i^{th} and j^{th} columns is denoted by C_{ij} or $C_i \leftrightarrow C_j$.

2. Multiplication of the elements of any row (or column) by a non-zero scalar quantity. The multiplication of i^{th} row by k is denoted by kR_i.
   Similarly, the multiplication of i^{th} column by k is denoted by kC_i.
3. Addition of constant multiplication of the elements of any row (or column) to the corresponding elements of any other row (or column).

3.3.1 Elementary Matrix

A matrix obtained from the unit matrix by applying any of the elementary transformation is called an elementary matrix.

e.g. Let
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply $R_2 \to R_2 + 3R_3$, then we have
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \text{ is an elementary matrix.}$$

3.4 ECHELON FORM OF A MATRIX

3.4.1 Row-Echelon Form of a Matrix

A matrix is said to be in row-echelon form if

a. All zero rows, if any, are at the bottom of the matrix.
b. First non-zero element of every row is on the right hand side of the first non-zero element in the preceeding row.

Note: The first non-zero element of any row is called key-element or pivotal element or pivot of that row.

3.4.2 Row Reduced Echelon Form of a Matrix

A matrix is said to be in row reduced echelon form if

a. It is in row echelon form.
b. Every key element is unity.
c. The elements above the key element in every column are all zero.

3.4.3 Column Echelon Form of a Matrix

A matrix is said to be in column echelon form if

a. All zero columns, if any, are at the extreme right of the matrix.
b. First non-zero element of every column is below the first non-zero element in the preceeding column.

Remark: The first non-zero element of any column is called the key-element or pivotal element or pivot of that column.

3.4.4 Column Reduced Echelon Form of a Matrix

A matrix is said to be in column reduced echelon form if

a. It is in column echelon form.
b. Every key element is unity.
c. The elements to the left of the key element in every row are all zero.

Remark: If A is row echelon form, then its transpose A' is in the column echelon form.

3.5 DETERMINANTS

The theory of determinant was originated from the study of system of linear equations. The determinant is a scalar value which is a function of the entries of a square matrix. The determinant of a square matrix $A = [a_{ij}]$ is denoted by det A or det (A) or $|A|$.

The determinant of a 1×1 (read as one cross one) matrix $A = [a]$ is denoted as $|A| = a$ and is called the determinant of order one.

The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and is called the determinant of order two.

Similarly, the determinant of a 3×3 matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is denoted as $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and is called the determinant of order 3.

3.5.1 Explanation of Determinant of Order Two (or Second Order)

Consider the following system of two linear equations having two unknowns x and y

$$a_1x + b_1y = 0 \quad \text{...(1)}$$

$$a_2x + b_2y = 0 \quad \text{...(2)}$$

Equation (1) gives

$$\frac{x}{y} = -\frac{b_1}{a_1} \quad \text{...(3)}$$

Equation (2) gives $$\frac{x}{y} = -\frac{b_2}{a_2} \quad \text{...(4)}$$

From equations (3) and (4), we can eliminate x and y to get

$$-\frac{b_1}{a_1} = -\frac{b_2}{a_2}$$

$$\Rightarrow \quad a_1b_2 - b_1a_2 = 0$$

The number $a_1b_2 - b_1a_2$ is called the determinant of order two.

The number $a_1b_2 - a_2b_1$ is also represented more conveniently by the symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

The numbers a_1, a_2, b_1, b_2 are called the elements of the determinant.

Remark: To expand a determinant of order 2 × 2, we apply the rule of cross-multiplication. The value of $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$.

SOME SOLVED EXAMPLES

Example 3.4. *Evaluate the determinant of* $\begin{bmatrix} 5 & -2 \\ 3 & 7 \end{bmatrix}$.

Solution. Let
$$A = \begin{bmatrix} 5 & -2 \\ 3 & 7 \end{bmatrix}$$

then
$$|A| = \begin{vmatrix} 5 & -2 \\ 3 & 7 \end{vmatrix}$$
$$= 35 + 6 = 41 \textbf{ Answer}$$

Example 3.5. *Find the value of x, if* $\begin{vmatrix} x & 3 \\ 6 & 2x \end{vmatrix} = \begin{vmatrix} 6 & 9 \\ 2 & 3 \end{vmatrix}$.

Solution. Given
$$\begin{vmatrix} x & 3 \\ 6 & 2x \end{vmatrix} = \begin{vmatrix} 6 & 9 \\ 2 & 3 \end{vmatrix}$$

then $2x^2 - 18 = 18 - 18$

$\Rightarrow$ $2x^2 - 18 = 0$

$\Rightarrow$ $2x^2 = 18$

$\Rightarrow$ $x^2 = 9$

$\Rightarrow$ $x = \pm 3$ **Answer**

3.5.2 Expansion of Determinant of Third Order

Consider the following system of three linear equations having three unknowns x, y and z.

$$a_1x + b_1y + c_1z = 0 \qquad ...(5)$$
$$a_2x + b_2y + c_2z = 0 \qquad ...(6)$$
$$a_3x + b_3y + c_3z = 0 \qquad ...(7)$$

To eliminate x, y, z from the above set of three equations, we solve (6) and (7)

$$\frac{x}{b_2c_3 - c_2b_3} = \frac{y}{c_2a_3 - a_2c_3} = \frac{z}{a_2b_3 - b_2a_3} = k \quad \text{(say)}$$

From here, we have
$$x = k(b_2c_3 - c_2b_3)$$
$$y = k(c_2a_3 - a_2c_3)$$
$$z = k(a_2b_3 - b_2a_3)$$

Substituting these values of x, y, z in equation (5), we get

$$a_1(b_2c_3 - c_2b_3)k + b_1(c_2a_3 - a_2c_3)k + c_1(a_2b_3 - b_2a_3)k = 0$$

or $\quad a_1(b_2c_3 - c_2b_3) + b_1(c_2a_3 - a_2c_3) + c_1(a_2b_3 - b_2a_3) = 0$ ...(8)

(Students think why k cann't be zero)

This expression on the left hand side of (8) is called the determinant of third order. Symbolically, it is written as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$\Rightarrow \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - c_2b_3) + b_1(c_2a_3 - a_2c_3) + c_1(a_2b_3 - b_2a_3)$$

$$\text{or} \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Remarks: i. The above expansion of determinant is known as expanding in terms of 1st row.

Similarly, the determinant can be expanded along any row or any column and in each case the value of the determinant remains the same.

ii. Sign before each term = $(-1)^{i+j}$, where 'i' and 'j' indicate the row and the column in which the element lie.

This is valid for determinant of any order.

SOME SOLVED EXAMPLES

Example 3.6. *Evaluate the determinant of* $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 31 & 11 & 38 \end{bmatrix}$.

Solution. Given $\quad A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 31 & 11 & 38 \end{bmatrix}$

then $\quad |A| = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 31 & 11 & 38 \end{vmatrix}$

Expanding along 1st row, we have

$$|A| = 1\begin{vmatrix} 6 & 10 \\ 11 & 38 \end{vmatrix} - 3\begin{vmatrix} 2 & 10 \\ 31 & 38 \end{vmatrix} + 5\begin{vmatrix} 2 & 6 \\ 31 & 11 \end{vmatrix}$$

$= 1(228 - 110) - 3(76 - 310) + 5(22 - 186)$

$= 1(118) - 3(-234) + 5(-164) = 118 + 702 - 820$

$= 0$ **Answer**

Example 3.7. *If* $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, *then show that* $|3A| = 27|A|$.

Solution. Given $$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

then $$3A = 3\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{bmatrix}$$

then the determinant of $|3A|$,

$$|3A| = \begin{vmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{vmatrix}$$

Expanding along 1st row, we have

$$|3A| = 3\begin{vmatrix} 3 & 6 \\ 0 & 12 \end{vmatrix} - 0\begin{vmatrix} 0 & 6 \\ 0 & 12 \end{vmatrix} + 3\begin{vmatrix} 0 & 3 \\ 0 & 0 \end{vmatrix}$$

$$= 3(36 - 0) - 0(0 - 0) + 3(0 - 0)$$

$$= 108 - 0 + 0$$

$$= 108 \quad \text{(L.H.S.)}$$

and $$|A| = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix}$$

Expanding along 1st row, we have

$$|A| = 1\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} - 0\begin{vmatrix} 0 & 2 \\ 0 & 4 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= 1(4) - 0 + 1(0)$$

$$= 4$$

then $$27|A| = 27 \times 4 = 108 \quad \text{(R.H.S.)}$$

Thus, we have, $$|3A| = 27|A|$$ **Proved**

3.5.3 Properties of Determinant

Property i. The value of the determinant remains unchanged if all it rows are changed into columns and all columns are changed into rows.

Example 3.8. *Evaluate* $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 6 \\ 7 & 8 & 9 \end{vmatrix}$.

Solution. Given, $$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Expanding along second row, we have

$$\Delta = 36 \qquad \text{(Students can find)}$$

Interchanging rows into columns, then we have

$$\text{(say) } \Delta_1 = \begin{vmatrix} 1 & 0 & 7 \\ 2 & 0 & 8 \\ 3 & 6 & 9 \end{vmatrix}$$

Expanding along first row, we have

$$\Delta_1 = 36 \qquad \text{(Students can verify it)}$$

Thus, $\Delta = \Delta_1$

Property ii. If any two rows or any two columns of a determinant are interchanged, then the sign of the value of the determinant is also changed.

Example 3.9. *Evaluate* $\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 6 \\ 1 & 2 & 3 \end{vmatrix}$.

Solution. Given, $$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 6 \\ 1 & 2 & 3 \end{vmatrix}$$

After expanding along 1st row, we have

$$\Delta = 2$$

Using Property ii, *i.e.* interchanging second and third rows, we have

$$\text{(say) } \Delta_1 = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 6 \end{vmatrix}$$

then expanding through 1st row, we have

$$\Delta_1 = -2$$

Thus, we have, $\Delta = -\Delta_1$

Property iii. If any two rows or two columns of a determinant are identical, then the value of the determinant is always zero.

Example 3.10. *Evaluate* $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 5 & 6 \end{vmatrix}$.

Solution. Given $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 5 & 6 \end{vmatrix}$

Expanding along 1st row, we have

$\Delta = 0$ (Students can calculate)

Thus, $\Delta = 0$

Property iv. If each element of any row or column of a determinant is multiplied by the same constant, then the value of the determinant is also multiplied by that factor.

Example 3.11. *Evaluate* $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{vmatrix}$.

Solution. Given $\Delta = \begin{vmatrix} 1 & 2 & 4 \\ 3 & 1 & 5 \\ 0 & 4 & 6 \end{vmatrix}$

Expanding through 1st row, we have

$$\Delta = -3$$

Multiply the 1st row by 5, we get

$$\text{(say) } \Delta_1 = \begin{vmatrix} 5 & 10 & 15 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{vmatrix}$$

$$= -15 = 5(\Delta)$$

$\Rightarrow$ $\Delta_1 = 5\Delta$

Property v. The value of the determinant remains unchanged if the elements of one row (or column) be added to any constant multiple of the corresponding elements of other row (or column) respectively.

Example 3.12. *Evaluate* $\Delta = \begin{vmatrix} 1 & 2 & 4 \\ 3 & 1 & 5 \\ 0 & 4 & 6 \end{vmatrix}$.

Solution. Given $\Delta = \begin{vmatrix} 1 & 2 & 4 \\ 3 & 1 & 5 \\ 0 & 4 & 6 \end{vmatrix}$

On expanding through 1st row, we have

$$\Delta = -2$$

On multiplying the second column by 3 and adding to the first column, we get

$$\text{(say) } \Delta_1 = \begin{vmatrix} 1+6 & 2 & 4 \\ 3+3 & 1 & 5 \\ 0+12 & 4 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} 7 & 2 & 4 \\ 6 & 1 & 5 \\ 12 & 4 & 6 \end{vmatrix}$$

On expanding along 1st row, we have

$$\Delta_1 = -2$$

Thus $\Delta = \Delta_1$

Property vi. If each element of a row (or column) of a determinant is expressed as a sum of two (or more) terms, then the determinant can also be expressed as the sum of two (or more) determinant.

Example 3.13. *Evaluate* $\Delta = \begin{vmatrix} 2+1 & 1 & 0 \\ 3+1 & 0 & 1 \\ 2+2 & 1 & 0 \end{vmatrix}$.

Solution. Given $\Delta = \begin{vmatrix} 2+1 & 1 & 0 \\ 3+1 & 0 & 1 \\ 2+2 & 1 & 0 \end{vmatrix}$

$$= \begin{vmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix}$$

$= 1$ (Students can calculate and verify)

> **Remark:** If A is a square matrix of order n, then $|kA| = k^n |A|$

Example 3.14. *Using Properties of determinant, prove that*

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca).$$

Solution. Let $\Delta = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$

Operating $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$, we have

$$\Delta = \begin{vmatrix} a & a^2 & bc \\ b-a & b^2-a^2 & ca-bc \\ c-a & c^2-a^2 & ab-bc \end{vmatrix}$$

$$\Delta = \begin{vmatrix} a & a^2 & bc \\ b-a & (b-a)(b+a) & c(a-b) \\ c-a & (c-a)(c+a) & b(a-c) \end{vmatrix}$$

Taking out $(b-a)$ and $(c-a)$ common from R_2 and R_3 respectively, we have

$$\Delta = (b-a)\,(c-a)\begin{vmatrix} a & a^2 & bc \\ 1 & a+b & -c \\ 1 & a+c & -b \end{vmatrix}$$

Operating $R_3 \to R_3 - R_2$, we have

$$\Delta = (b-a)\,(c-a)\begin{vmatrix} a & a^2 & bc \\ 1 & a+b & -c \\ 0 & c-b & c-b \end{vmatrix}$$

Taking out $(c-b)$ common from R_3, we have

$$= (b-a)\,(c-a)\,(c-b)\begin{vmatrix} a & a^2 & bc \\ 1 & a+b & -c \\ 0 & 1 & 1 \end{vmatrix}$$

Operating through third row, we have

$$= [(b-a)\,(c-a)\,(c-b)]\,(-1)\begin{vmatrix} a & bc-a^2 \\ 1 & -c-a-b \end{vmatrix}$$

$$= [(b-a)\,(c-a)\,(c-b)]\,(-1)\,[-ac-a^2-ab-bc+a^2]$$

$$= [(b-a)\,(c-a)\,(c-b)]\,(-1)\,[(-1)\,(ac+ab+bc)]$$

$$= (a-b)\,(b-c)\,(c-a)\,(ab+bc+ca)$$ **Proved**

Example 3.15. *Show that* $\Delta = \begin{vmatrix} 1 & a & abc \\ 1 & b & bca \\ 1 & c & cab \end{vmatrix} = 0.$

Solution. Let $\Delta = \begin{vmatrix} 1 & a & abc \\ 1 & b & bca \\ 1 & c & cab \end{vmatrix}$

Taking out common abc from c_3, we get

$$\Delta = abc\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}$$

$= 0$ [As 1st and 3rd column are same] [As per Property (iii)]

Example 3.16. *Using Property of determinant, prove that* $\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = (x+2)\,(x-1)^2.$

Solution. Given $\Delta = \begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$

Operating $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} x+2 & 1 & 1 \\ x+2 & x & 1 \\ x+2 & 1 & x \end{vmatrix}$$

Taking out $(x + 2)$ common from C_1, we get

$$= (x+2) \begin{vmatrix} 1 & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$$

Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have

$$\Delta = (x+2) \begin{vmatrix} 1 & 1 & 1 \\ 0 & (x-1) & 0 \\ 0 & 0 & (x-1) \end{vmatrix}$$

Taking out $(x - 1)$ common from R_2 and R_3 respectively, we have

$$\Delta = (x+2)\,(x-1)^2 \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along R_2, we have

$$\Delta = (x+2)\,(x-1)^2 \,.\, (1) \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= (x+2)\,(x-1)^2 \,.\, (1)$$

$$= (x+2)\,(x-1)^2$$ **Proved**

EXERCISE 3.3

1. Evaluate the following determinant:

a. $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

b. $\begin{vmatrix} \sqrt{6} & \sqrt{5} \\ \sqrt{20} & \sqrt{24} \end{vmatrix}$

c. $\begin{vmatrix} 210 & 117 & 345 \\ 19 & 9 & 34 \\ 7 & 3 & 5 \end{vmatrix}$

d. $\begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$

2. Evaluate the given determinant $\Delta = \begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

a. with the help of second row

b. with the help of third column

3. Find the value of the given determinant $\Delta = \begin{vmatrix} 0 & 1 & \sec\theta \\ \tan\theta & -\sec\theta & \tan\theta \\ 1 & 1 & 1 \end{vmatrix}$.

4. Using the Properties of determinant, prove the following:

a. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (b - c)\,(c - a)\,(a - b)\,(a + b + c)$

b. $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3$

c. $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1 - x^3)^2$

d. $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

e. $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2(a + b + c)\,(ab + bc + ca - a^2 - b^2 - c^2)$

5. Show that $x = 2$ is a root of the given equation $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$ and solve it completely.

6. If a, b and c are real and $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$, then show that either $a + b + c = 0$ or $a = b = c$.

Answers

1. a. $x^3 - x^2 + 2$ b. 2 c. 2691 d. 40

2. a. 23 b. 23 **3.** $\sec\theta\,(\sec\theta + \tan\theta)$

3.5.4 Applications of Determinants

3.5.4.1 Area of Triangle by using Determinant

The area of triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\text{Area} = \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$$

or $$\frac{1}{2}\left[x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3\right]$$

This expression in the form of determinant can be written as

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{or} \quad \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

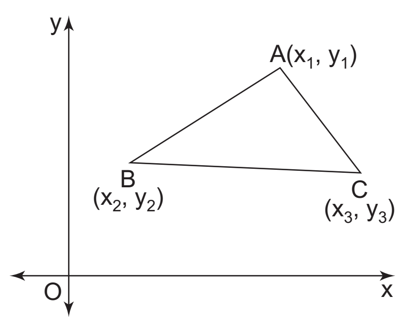


Fig. 3.2

Note: 1. Determinant could be negative but area is always non-negative *i.e.*, ≥ 0.
2. If the area of a triangle is given, then use positive as well as negative values for calculations.
3. Three points are **collinear** if and only if area of triangle formed by three points is zero.

SOME SOLVED EXAMPLES

Example 3.17. *Find the area of the triangle whose vertices are (2, 7), (1, 1), (10, 8).*

Solution. Area of triangle is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Vertices of triangle are $(\underset{x_1}{2}, \underset{y_1}{7}), (\underset{x_2}{1}, \underset{y_2}{1}), (\underset{x_3}{10}, \underset{y_3}{8})$

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 10 \\ 7 & 1 & 8 \end{vmatrix}$$

Expanding about R_1, $\Delta = \frac{1}{2}[1(8-10) - 1(16-70) + 1(2-7)]$

$$= \frac{1}{2}[1(-2) - 1(-54) + 1(-5)]$$

$$= \frac{1}{2}[-2 + 54 - 5] = \frac{47}{2}$$

Thus, required area of triangle is $\frac{47}{2}$ square unit.

Example 3.18. *Find the area of triangle whose vertices are (–2, –3), (3, 2), (–1, –8).*

Solution. Vertices of triangle are $(\underset{x_1}{-2}, \underset{y_1}{-3}), (\underset{x_2}{3}, \underset{y_2}{2}), (\underset{x_3}{-1}, \underset{y_3}{-8})$

Area of triangle is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ -2 & 3 & -1 \\ -3 & 2 & -8 \end{vmatrix}$$

$$= \frac{1}{2}[1(-24+2) - 1(16-3) + 1(-4+9)] \qquad \text{[Expanding about } R_1\text{]}$$

$$= \frac{1}{2}[-22 - 13 + 5] = -\frac{30}{2} = -15$$

Since area of triangle is always non-negative, therefore, the required area of triangle is 15 square units.

Example 3.19. *Find the value of k if area of triangle is 4 square units and vertices are (–2, 0), (0, 4), (0, k).*

Solution. Area of the triangle is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Here, area of triangle = 4 square units

$\therefore \quad \Delta = \pm 4$

[area is non-negative but determinant can be positive and negative]

Putting $x_1 = -2, x_2 = 0, x_3 = 0$ and $y_1 = 0, y_2 = 4, y_3 = k$

$$\pm 4 = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ -2 & 0 & 0 \\ 0 & 4 & k \end{vmatrix}$$

$$\pm 4 = \frac{1}{2}\left[-(-2)(k-4)\right] \qquad \text{[Expanding about } R_2\text{]}$$

$$\pm 8 = 2(k-4)$$

$$\pm 4 = k-4$$

So, $4 = k - 4$ and $-4 = k - 4$

$4 = k - 4$	$-4 = k - 4$
$4 + 4 = k$	$-4 + 4 = k$
$8 = k$	$0 = k$
$\therefore \; k = 8$	$\therefore \; k = 0$

Required value of $k = 8, 0$.

Example 3.20. *Show that the points (–2, –1), (7, 8), (–3, –2) are collinear.*

Solution. Three points are collinear if they lie on same line.

$\therefore$ Area of triangle = 0

Area of triangle is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \qquad ...(1)$$

Points are $(\underset{x_1}{-2}, \underset{y_1}{-1}), (\underset{x_2}{7}, \underset{y_2}{8}), (\underset{x_3}{-3}, \underset{y_3}{-2})$

Putting the values in (1), we get

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ -2 & 7 & -3 \\ -1 & 8 & -2 \end{vmatrix}$$

Expanding about R_1, $\Delta = \frac{1}{2}[1(-14+24)-1(4-3)+1(-16+7)]$

$= \frac{1}{2}[10-1-9]=0$

So, $\Delta = 0$

Hence, given points are collinear.

EXERCISE 3.4

1. Find the area of the triangle with vertices at the point given in each of the following:

a. (1, 0), (6, 0), (4, 3) b. (3, 8), (–4, 2), (5, 1)

2. Find the value of k in following if

i. area of triangle is 4 sq. units and vertices are $(k, 0)$, (4, 0), (0, 2).

ii. area of triangle is 35 sq. units and vertices are (2, –6), (5, 4), $(k, 4)$.

3. Show that points $A(a, b + c)$, $B(b, c + a)$, $C(c, a + b)$ are collinear.

Answers

1. a. 15/2 b. 61/2 **2.** i. 0, 8 ii. 12, –2

3.5.5 Minors and co-factors

3.5.5.1 Minors

Let $A = [a_{ij}]$ be a square matrix of order n such that $|A| = |a_{ij}|$.

Then the minor of an element of the matrix A is defined as the determinant obtained by deleting the row and the column in which the element lies. Minors are required for calculating matrix cofactors.

Consider a matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

Thus, the minors of a_1, b_1 and c_1 are respectively, $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$.

Similarly, the minors of a_2, b_2 and c_2 are respectively, $\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}, \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$

Similarly, the minors of a_3, b_3 and c_3 are respectively, $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

3.5.5.2 Cofactor

The cofactor of any element of ith row and jth column is

$$\text{Cofactor} = (-1)^{i+j} \text{ minor}$$

Cofactors are useful for computing both the determinants and inverse of square matrices.

SOME SOLVED EXAMPLES

Example 3.21. *Find all the minors and cofactors of the matrix* $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 7 & 0 & -1 \end{bmatrix}$.

Solution. Given $$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 7 & 0 & -1 \end{bmatrix}$$

Here, $$a_{11} = 1, \quad a_{12} = 2, \quad a_{13} = 3$$
$$a_{21} = 4, \quad a_{22} = 3, \quad a_{23} = 2$$
$$a_{31} = 7, \quad a_{32} = 0, \quad a_{33} = -1$$

$$M_{11} = \text{minor of } a_{11} = \begin{vmatrix} 3 & 2 \\ 0 & -1 \end{vmatrix} = -3$$

$$M_{12} = \text{minor of } a_{12} = \begin{vmatrix} 4 & 2 \\ 7 & -1 \end{vmatrix} = -4 - 14 = -18$$

$$M_{13} = \text{minor of } a_{13} = \begin{vmatrix} 4 & 3 \\ 7 & 0 \end{vmatrix} = 0 - 21 = -21$$

$$M_{21} = \text{minor of } a_{21} = \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} = -2 - 0 = -2$$

$$M_{22} = \text{minor of } a_{22} = \begin{vmatrix} 1 & 3 \\ 7 & -1 \end{vmatrix} = -1 - 21 = -22$$

$$M_{23} = \text{minor of } a_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 0 \end{vmatrix} = 0 - 14 = -14$$

$$M_{31} = \text{minor of } a_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 4 - 9 = -5$$

$$M_{32} = \text{minor of } a_{32} = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = 2 - 12 = -10$$

$$M_{33} = \text{minor of } a_{33} = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = 3 - 8 = -5$$

$$A_{11} = \text{cofactor of } a_{11} = (-1)^{1+1} M_{11} = (-1)^2 (-3) = -3$$
$$A_{12} = \text{cofactor of } a_{12} = (-1)^{1+2} M_{12} = (-1)^3 (-18) = 18$$
$$A_{13} = \text{cofactor of } a_{13} = (-1)^{1+3} M_{13} = (-1)^4 (-21) = -21$$
$$A_{21} = \text{cofactor of } a_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-2) = 2$$
$$A_{22} = \text{cofactor of } a_{22} = (-1)^{2+2} M_{22} = (-1)^4 (-22) = -22$$
$$A_{23} = \text{cofactor of } a_{23} = (-1)^{2+3} M_{23} = (-1)^5 (-14) = 14$$

$$A_{31} = \text{cofactor of } a_{31} = (-1)^{3+1} M_{31} = (-1)^4 (-5) = -5$$
$$A_{32} = \text{cofactor of } a_{32} = (-1)^{3+2} M_{32} = (-1)^5 (-10) = 10$$
$$A_{33} = \text{cofactor of } a_{33} = (-1)^{3+3} M_{33} = (-1)^6 (-5) = -5$$

3.5.6 Adjoint of a Square Matrix

Adjoint of a square matrix A is obtained by replacing each element of A by its cofactor in $|A|$ and then taking the transpose of the matrix so obtained.

Let the determinant of a square matrix A be $|A|$.

Thus, if $$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}, \text{ then}$$

$$|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The matrix formed by the cofactors of the elements in $|A|$ is given by,

$$\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

where, $$A_1 = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} = (b_2c_3 - c_2b_3)$$

$$A_2 = -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} = -(b_1c_3 - c_1b_3)$$

$$A_3 = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = (b_1c_2 - b_2c_1)$$

$$B_1 = -\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} = -(a_2c_3 - c_2a_3)$$

$$B_2 = \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} = (a_1c_3 - c_1a_3)$$

$$B_3 = -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} = -(a_1c_2 - c_1a_2)$$

$$C_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_2b_3 - b_2a_3$$

$$C_2 = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = -(a_1b_3 - b_1a_3)$$

$$C_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = (a_1b_2 - b_1a_2)$$

Then, the transpose of the matrix of cofactors is $\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$ is called the adjoint of the matrix A and is denoted as adj. A.

3.5.6.1 Property of Adjoint

The product of a matrix A and its adjoint is equal to the unit matrix multiplied by the determinant of A.

Symbolically, if A is a square matrix, then

$$\text{adj. }(A)\,.\,A = A\,(\text{adj. }A) = |\,A\,|\,.\,I$$

where I is a unit matrix.

SOME SOLVED EXAMPLES

Example 3.22. *Find the adjoint of the given matrix* $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$.

Solution. Given $\quad A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

Here, $\quad a_{11} = 2, a_{12} = 3, a_{21} = 3, a_{22} = 5$

Cofactors of a_{11}, a_{12}, a_{21} and a_{22} is given by

$$A_{11} = \text{cofactor of } a_{11} = (-1)^{1+1}\,(5) = 5$$
$$A_{12} = \text{cofactor of } a_{12} = (-1)^{1+2}\,(3) = -3$$
$$A_{21} = \text{cofactor of } a_{21} = (-1)^{2+1}\,(3) = -3$$
$$A_{22} = \text{cofactor of } a_{22} = (-1)^{2+2}\,(2) = 2$$

Thus, $\quad \text{adj. }(A) = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}$

$$= \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \quad \textbf{Answer}$$

Example 3.23. *Find the adjoint of the given matrix* $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$.

Solution. Given $\quad A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

Here, $\quad a_{11} = 1, a_{12} = 2, a_{13} = 4$

$$a_{21} = 2, a_{22} = 3, a_{23} = 2$$
$$a_{31} = 3, a_{32} = 3, a_{33} = 4$$

Cofactors of $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}$ and a_{33} are calculate as discussed earlier.

Then, we have, $A_{11} = 6, \quad A_{12} = -2, \quad A_{13} = -3$

$A_{21} = 4, \quad A_{22} = -8, \quad A_{23} = 3$

$A_{31} = -8, \quad A_{32} = 6, \quad A_{33} = -1$

Thus, $$\text{adj. }(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$\Rightarrow$ $$\text{adj. }(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$\Rightarrow$ $$\text{adj. }(A) = \begin{bmatrix} 6 & 4 & -8 \\ -2 & -8 & 6 \\ -3 & 3 & -1 \end{bmatrix}$$ **Answer**

3.5.6.2 Inverse of a Matrix

If A and B are two square matrices of the same order, such that $AB = BA = I$, then B is called the inverse of A *i.e.* $B = A^{-1}$ and 'A' is the inverse of B.

Remarks: 1. Condition for a square matrix 'A' to possess an inverse is that matrix 'A' should be non-singular *i.e.* $|A| \neq 0$.

2. Any square matrix which possess an inverse is called an invertible matrix.

3. If any square matrix possess inverse, then it is always unique.

To find the inverse of a matrix 'A' with the help of its adjoint matrix, we have

$$A^{-1} = \frac{\text{adj. }(A)}{|A|} = \frac{1}{|A|}(\text{Adj. } A)$$

(Students can find this result with the help of property of adjoint discussed earlier.)

SOME SOLVED EXAMPLES

Example 3.24. *Find the inverse of the matrix* $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$.

Solution. Given $A = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 10 - 9 = 1 \neq 0$

$\Rightarrow$ As $|A| \neq 0$, so 'A' is non-singular and hence A^{-1} exists.

So, we need to find cofactors.

Thus, A_{11} = cofactor of $a_{11} = 5$

A_{12} = cofactor of $a_{12} = -3$

A_{21} = cofactor of $a_{31} = -3$

A_{22} = cofactor of $a_{22} = 2$

$$\Rightarrow \qquad \text{Adj.}(A) = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$$

Now, $$A^{-1} = \frac{\text{Adj.}(A)}{|A|} = \frac{1}{1}\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$$

$$\Rightarrow \qquad A^{-1} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$$ **Answer**

Example 3.25. *Find the inverse of the matrix* $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$ *and also verify that* $AA^{-1} = A^{-1}A = I$.

Solution. Given $$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$$

So we have, $$|A| = \begin{vmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{vmatrix} = 1 \neq 0$$

Thus $|A| \neq 0$ means 'A' is non-singular matrix and hence A^{-1} exists,

To find A^{-1}, we have to find Adjoint of A, for this we will find cofactors of A.

We have $A_{11} = -9,\quad A_{12} = -8,\quad A_{13} = -2$

$A_{21} = 8,\quad A_{22} = 7,\quad A_{23} = 2$

$A_{31} = -5,\quad A_{32} = -4,\quad A_{33} = -1$

$$\therefore \qquad \text{Adj.}(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & -1 \end{bmatrix}$$

Hence, $$A^{-1} = \frac{1}{|A|}\text{Adj.}(A)$$

$$= \begin{bmatrix} -9 & 8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & -1 \end{bmatrix}$$

For verification, $AA^{-1} = A^{-1}A = I$ (Students can try by themselves.)

EXERCISE 3.5

Questions Based on Minors, Cofactors, Adjoint and Inverse using Adjoint for a given Matrix

1. Find all the minors and cofactors of each element for the given determinant. Also solve the given determinant $A = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$.

2. Find the adjoint of the given matrices:

a. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$

3. Find the adjoint of the given matrix $A = \begin{bmatrix} 3 & 5 & 7 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

4. Find the inverse of the following matrices:

a. $\begin{bmatrix} 1 & 2 & 3 \\ -3 & 5 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 3 & 5 \\ 2 & 7 \end{bmatrix}$

5. Find the inverse of the given matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ -2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$.

6. Given $D = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$ where none d_1, d_2, d_3 and d_4 are zero. Find D^{-1}.

Answers

1. $M_{11} = (ab^2 - ac^2)$, $M_{12} = (ab - ac)$, $M_{13} = (c - b)$
$M_{21} = a^2b - bc^2$, $M_{22} = (ab - bc)$, $M_{23} = (c - a)$
$M_{31} = (ca^2 - cb^2$, $M_{32} = (ca - bc)$, $M_{33} = (b - a)$
$A_{11} = (ab^2 - ac^2)$, $A_{12} = (ac - ab)$, $A_{13} = (c - b)$
$A_{21} = (bc^2 - a^2b)$, $A_{22} = (ab - bc)$, $A_{23} = (a - c)$
$A_{31} = (ca^2 - cb^2)$, $A_{32} = (bc - ca)$, $A_{33} = (b - a)$, $|A| = (a - b)(b - c)(c - a)$

2. a. $\begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$ b. $\begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$ **3.** $\begin{bmatrix} -7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 2 & -19 \end{bmatrix}$

4. a. $\frac{1}{2}\begin{bmatrix} 5 & 1 & -15 \\ 3 & 1 & -9 \\ -3 & -1 & 11 \end{bmatrix}$ b. $\frac{1}{11}\begin{bmatrix} 7 & -5 \\ -2 & 3 \end{bmatrix}$

5. $\frac{1}{5}\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -20 \\ 2 & 3 & -15 \end{bmatrix}$ **6.** $\begin{bmatrix} 1/d_1 & 0 & 0 & 0 \\ 0 & 1/d_2 & 0 & 0 \\ 0 & 0 & 1/d_3 & 0 \\ 0 & 0 & 0 & 1/d_4 \end{bmatrix}$

INTERESTING FACTS

- The structure of various buildings can be changed or designed with the help of matrices. We can take an example of "***Burj Khalifa***". The design which is not so common was made by using matrices.
- Some specially designed functions, such as the ***Iterated Function Systems***, are really fun to draw and are computed with the use of matrices.
- In the domain of IT and Information Security (especially encryption), many IT companies also use these matrices as data structures to perform search queries, track user information, and manage databases.
- It can also be used at many places, such as the matrix rotating while playing the ***car racing game***; building a cluster of networks over the ***Face book, Twitter, Instagram***; or even while trading in the ***Wall Street***.

VIDEO REFERENCES

Basic Matrix Concepts

Introduction to Matrix Algebra - I

Matrix Analysis with Applications

Elementary Row Operations

Determinant of a Matrix

APPLICATIONS TO REAL LIFE

- Various Graphic software such as Adobe Photoshop on your personal computer uses matrices to process linear transformations to render images.
- A square matrix can represent a linear transformation of a geometric object, for example, a matrix reflects an object in x or y-axis in a cartesian x-y plane.
- It also has its application in the domain of gaming industry and image processing domain, where reflections in ponds, rivers, and other upside-down images are seen.

- Matrices also play an important role in computer graphics, like when people want to apply any of the desired matrix transformations on any object, for example cartoon characters.
- They also play significant role in plotting surveys, representation of real-world data such as the population of people, infant mortality rate can be done through them.
- Even in economics, constructing the predictive model of dependent variables, analysing the shares, studying the trends of business can be done through matrices.
- In physics, while calculating the battery power outputs, solving Kirchhoff's Law, and in the field of quantum physics, matrix does play an important part.
- Also in geology, they play a crucial part while making seismic surveys.
- In robotics, the robotic movements are defined on the basis of matrices.

3.6 RANK OF A MATRIX

A non-negative number 'r' is said to be the rank of a matrix 'A' if

1. There exist atleast one minor (square submatrix) of order 'r' which is not zero.
2. Every minor of matrix 'A' of order greater than 'r' is zero. Rank of A is denoted by $\rho(A)$.

Remarks: 1. $\rho(A) = 0$, when A is a zero matrix.
2. $\rho(A) \geq 1$, when $A \neq 0$.
3. If 'A' is a non-singular matrix of order 'n', then $\rho(A) = n$.
4. If 'A' is a singular matrix of order n, then $\rho(A) < n$.
5. If the order of a matrix 'A' is $m \times n$, then $\rho(A) \leq \min.(m, n)$.
6. Corresponding to every matrix 'A' of rank 'r', there exist non-singular matrices P and Q such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

For example: *Find the rank of the matrix* $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution. Given $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

then $|A| = 2(1) = 2 \neq 0$

$\Rightarrow$ A is a non-singular matrix of order 3

$\Rightarrow$ $\rho(A) = 3$.

3.6.1 Another Way to Find the Rank of a Matrix

The rank of a matrix is equal to the number of non-zero rows in Echelon Form of that Matrix.

Remark: Non-zero row is that row in which atleast one element is not zero.

e.g. Consider a matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Clearly, the given matrix 'A' is in Echelon form, which has two non-zero rows.
Hence the rank of $A = 2$ *i.e.* $\rho(A) = 2$.

SOME SOLVED EXAMPLES

Example 3.26. *Find the rank of the matrix* $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$.

Solution. We have,
$$A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1, R_3 \to R_3 - R_1$
$$A \sim \begin{bmatrix} -1 & 2 & -2 \\ 0 & 4 & -1 \\ 0 & -3 & 2 \end{bmatrix}$$

$R_3 \to R_3 + R_2$
$$A \sim \begin{bmatrix} -1 & 2 & -2 \\ 0 & 4 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_3 \to 4R_3 - R_2$
$$A \sim \begin{bmatrix} -1 & 2 & -2 \\ 0 & 4 & -1 \\ 0 & 0 & 5 \end{bmatrix}$$

The number of non-zero row is 3, therefore Rank $(A) = 3$

Example 3.27. *Find the rank of the matrix* $A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ -2 & 3 & 7 & -1 \\ 1 & 9 & 16 & -13 \end{bmatrix}$.

Solution. Here, we have,
$$A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ -2 & 3 & 7 & -1 \\ 1 & 9 & 16 & -13 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1, R_3 \to R_3 - R_1$
$$A \sim \begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 7 & 13 & -9 \\ 0 & 7 & 13 & -9 \end{bmatrix}$$

$R_3 \to R_3 - R_2 \qquad A \sim \begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 7 & 13 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The number of non-zero row is 2, therefore Rank $(A) = 2$.

Example 3.28. *For which value of 'b' the rank of the matrix* $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{bmatrix}$ *is 2.*

Solution. Here we have, $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{bmatrix}$

$R_1 \to 2R_1 \qquad A \sim \begin{bmatrix} 2 & 10 & 8 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{bmatrix}$

$R_3 \to R_3 - R_1 \qquad A \sim \begin{bmatrix} 2 & 10 & 8 \\ 0 & 3 & 2 \\ b-2 & 3 & 2 \end{bmatrix}$

$R_3 \to R_3 - R_2 \qquad A \sim \begin{bmatrix} 2 & 10 & 8 \\ 0 & 3 & 2 \\ b-2 & 0 & 0 \end{bmatrix}$

If rank of A is 2, then $b - 2$ must be zero

i.e., $\qquad b - 2 = 0 \quad \Rightarrow \quad b = 2.$

Example 3.29. *Find the rank of the matrix* $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$.

Solution. Here we have, $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$

$R_2 \to R_2 - \frac{3}{2}R_1,\ R_3 \to R_3 - 2R_1,\ R_4 \to R_4 - \frac{9}{2}R_1$

$$A \sim \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & \frac{-1}{2} & -1 & \frac{-3}{2} \\ 0 & -1 & -2 & -3 \\ 0 & \frac{-7}{2} & -7 & \frac{-21}{2} \end{bmatrix}$$

$R_3 \to R_3 - 2R_2, R_4 \to R_4 - 7R_2$

$$A \sim \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & \frac{-1}{2} & -1 & \frac{-3}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, the number of non-zero row is 2 therefore Rank $(A) = 2$.

Example 3.30. *Find the rank of the matrix* $A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix}$.

Solution. We have, $$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 - 6R_1$

$$A \sim \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & -7 & -5 & -2 \\ 0 & -14 & -10 & -4 \\ 0 & -21 & -16 & -6 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2, R_4 \to R_4 - 3R_2$

$$A \sim \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & -7 & -5 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$ $$A \sim \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & -7 & -5 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, the number of non-zero row is 3, therefore Rank $(A) = 3$.

Example 3.31. *Find the rank of the matrix* $A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$.

Solution. Here, we have, $$A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_1 \leftrightarrow R_2 \qquad A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & 1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 - 6R_1$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 9 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_3 \to R_3 - \frac{4}{5}R_2, R_4 \to R_4 - \frac{9}{5}R_2$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 9 \\ 0 & 0 & 33/5 & 14/5 \\ 0 & 0 & 33/5 & 4/5 \end{bmatrix} \Rightarrow R_4 \to R_4 - R_3, \quad A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 9 \\ 0 & 0 & 33/5 & 14/5 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Rank = number of non-zero rows = 4, Rank (A) = 4.

Example 3.32. *Find the rank of the matrix* $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$.

Solution. Here we have, $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 - 6R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_4 \to R_4 - R_3 \qquad A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$

$R_4 \to R_4 - R_2 \qquad A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow R_2 \leftrightarrow R_3, \quad A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Rank = number of non-zero rows = 3, Rank (A) = 3.

3.7 NORMAL FORM OF A MATRIX (CANONICAL FORM)

Every non-zero matrix of order $m \times n$ can be reduced by means of elementary row and column operation into equivalent matrix of any of the following forms:

i. $\begin{bmatrix} I_r & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 0 \end{bmatrix}$ *ii.* $\begin{bmatrix} I_r \\ \cdots\cdots \\ 0 \end{bmatrix}$

iii. $[I_r \;:\; 0]$ iv. $[I_r]$

Where I_r is the identity matrix of order r and 0 represent zero matrix of any order which is called its normal form or canonical form. The number r so obtained is called the rank of A and we write $\rho(A) = r$. The form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is called the first canonical form of A. Since both row and column transformation may be used here, the element 1 of the first row obtained can be moved in first column. Then both the first row and first column can be cleared of other non-zero elements. Similarly, the element 1 of the second row can be brought into the second column and so on.

SOME SOLVED EXAMPLES

Example 3.33. *Find the rank of the matrix* $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & -4 & 4 & -7 \\ 1 & 2 & 1 & 2 \end{bmatrix}$ *by reducing it to normal form.*

Solution. Let
$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & -4 & 4 & -7 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1, C_3 \to C_3 + C_1, C_4 \to C_4 - 3C_1$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 2 & -1 \\ 1 & 0 & 2 & -1 \end{bmatrix}$$

$C_3 \to C_3 + 2C_4, C_4 \to (-1)C_4$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1, R_3 \to R_3 - R_1$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \leftrightarrow C_4$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_2 & : & 0 \\ 0 & : & 0 \end{bmatrix}$

Which is required normal form.
Rank of Matrix $(A) = 2$.

Example 3.34. *Find the rank of the matrix* $A = \begin{bmatrix} 9 & 0 & 2 & 3 \\ 0 & 1 & 5 & 6 \\ 4 & 5 & 3 & 0 \end{bmatrix}$ *by reducing it to normal form.*

Solution. We have $\qquad A = \begin{bmatrix} 9 & 0 & 2 & 3 \\ 0 & 1 & 5 & 6 \\ 4 & 5 & 3 & 0 \end{bmatrix}$

$R_1 \to R_1\,(1/9)$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 2/9 & 3/9 \\ 0 & 1 & 5 & 6 \\ 4 & 5 & 3 & 0 \end{bmatrix}$

$R_3 \to R_3 + R_1(-4)$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 2/9 & 3/9 \\ 0 & 1 & 5 & 6 \\ 0 & 5 & 19/9 & -12/9 \end{bmatrix}$

$C_3 \to C_3 + C_1\left(\frac{-2}{9}\right)$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 3/9 \\ 0 & 1 & 5 & 6 \\ 0 & 5 & 19/9 & -12/9 \end{bmatrix}$

$C_4 \to C_4 + \left(\frac{-3}{9}\right)C_1$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 6 \\ 0 & 5 & 19/9 & -12/9 \end{bmatrix}$

$R_3 \to R_3 - R_2(5)$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & \frac{-206}{9} & \frac{-282}{9} \end{bmatrix}$

$C_3 \to C_3 + C_2(-5)$
$C_4 \to C_4 + C_2(-6)$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-206}{9} & \frac{-282}{9} \end{bmatrix}$

$R_3 \to R_3\left(\frac{-9}{206}\right)$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{141}{103} \end{bmatrix}$

$$C_4 \to C_4 + C_3\left(\frac{-141}{103}\right) \qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = [I_3 : 0]$$

Which is required normal form.
Rank of Matrix $(A) = 3$.

Example 3.35. *Reduce the matrix* $A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$ *to normal form* $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

Solution. We have $\quad A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

$C_2 \to C_2 + C_1, C_3 \to C_3 - 2C_1, C_4 \to C_4 + 3C_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 5 & -8 & 14 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$R_2 \to R_2 - 4R_1 \qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & -8 & 14 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$R_4 \leftrightarrow R_2 \qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 5 & -8 & 14 \end{bmatrix}$$

$$C_4 \to C_4 - 2C_2 \qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & -2 \\ 0 & 5 & -8 & 4 \end{bmatrix}$$

$$\begin{matrix} R_3 \to R_3 - 3R_2 \\ R_4 \to R_4 - 5R_2 \end{matrix} \qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -8 & 4 \end{bmatrix}$$

$$C_3 \leftrightarrow C_4 \qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 4 & -8 \end{bmatrix}$$

$C_3 \to \frac{-1}{2} C_3$, $C_4 \to \frac{-1}{8} C_4$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}$

$R_4 \to R_4 + 2R_3$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$

Hence $\rho(A) = 4$.

Example 3.36. *Find the rank of the following matrix by reducing it to normal form* $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$.

Solution. We have $\quad A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

$R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - R_1$ $\quad A \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

$C_2 \to C_2 - 2C_1$, $C_3 \to C_3 + C_1$, $C_4 \to C_4 - 3C_1$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

$C_3 \to C_3 + \frac{6}{7} C_2$, $C_4 \to C_4 - \frac{11}{7} C_2$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

$R_4 \to R_4 + \frac{1}{2} R_3$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_4 \to C_4 + 2C_3$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_2 \to \frac{-1}{7} R_2$

$R_3 \to \frac{-1}{2} R_3$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$

Rank of $A = 3$.

Example 3.37. *Reduce the matrix to normal form and find its rank if* $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$.

Solution. We have $\quad A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$

$R_2 \to R_2 - \frac{3}{2} R_1$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - \frac{9}{2} R_1$ $\quad A \sim \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & -1/2 & -1 & -3/2 \\ 0 & -1 & -2 & -3 \\ 0 & -7/2 & -7 & -21/2 \end{bmatrix}$

$C_2 \to C_2 - \frac{3}{2} C_1$

$C_3 \to C_3 - 2C_1$

$C_4 \to C_4 - \frac{5}{2} C_1$ $\quad A \sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1/2 & -1 & -3/2 \\ 0 & -1 & -2 & -3 \\ 0 & -7/2 & -7 & -21/2 \end{bmatrix}$

$R_1 \to \frac{1}{2} R_1$

$R_2 \to -2R_2$

$R_3 \to -R_3$

$R_4 \to \frac{-2}{7} R_4$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

$R_3 \to R_3 - R_2$

$R_4 \to R_4 - R_2$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_3 \to C_3 - 2C_2$
$C_4 \to C_4 - 3C_2$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence its rank = 2.

Example 3.38. *Reduce the matrix A to its normal form, when* $A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$.

Hence find the rank of A.

Solution. The given matrix is $A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 + R_1$
$$A \sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix} \Rightarrow \begin{matrix} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 + C_1 \\ C_4 \to C_4 - 4C_1 \end{matrix} \quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}$$

$C_3 \leftrightarrow C_2$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & -4 \\ 0 & 4 & 0 & 0 \\ 0 & 5 & 0 & -3 \end{bmatrix} \Rightarrow \begin{matrix} R_3 \to R_3 - \frac{4}{5}R_2 \\ R_4 \to R_4 - R_2 \end{matrix} \quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & -4 \\ 0 & 0 & 0 & 16/5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to 1/5R_2$
$R_3 \to 5/16R_3$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4/5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{matrix} R_2 \to R_2 + \frac{5}{4}R_3 \\ R_4 \to R_4 - R_3 \end{matrix} \quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \leftrightarrow C_4$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Which is required normal form

Hence the rank of given matrix is = 3.

3.7.1 To Calculate *P* and *Q* where $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

If A is a matrix of order $m \times n$ then write $A = I_m A I_n$ where I_m and I_n are mth and nth order unit matrices respectively. The elementary row operation on A can be effected by premultiplication with corresponding elementary matrix *i.e.* application of the same to I_m or to the matrix obtained from I_m in subsequent steps. Similarly, application of an elementary column operation to A is equivalent to application of the same to I_n or the matrix obtained from I_n in subsequent steps. In the end when we get $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ in place of A on left hand side, we have P in place of I_m and Q in place of I_n on the right hand side.

SOME SOLVED EXAMPLES

Example 3.39. *Find the non-singular matrices P and Q such that PAQ is in normal form where* $A = \begin{bmatrix} 2 & 2 & -6 \\ -1 & 2 & 2 \end{bmatrix}$.

Solution. We have $[A]_{2\times 3} = I_2 \,.\, A \,.\, I_3$

$$\begin{bmatrix} 2 & 2 & -6 \\ -1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to \frac{1}{2} R_1$
$$\begin{bmatrix} 1 & 1 & -3 \\ -1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 + R_1$
$$\begin{bmatrix} 1 & 1 & -3 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - C_1, C_3 \to C_3 + 3C_1$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to \frac{1}{3} C_2$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1/3 & 3 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 + C_2$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1/3 & 8/3 \\ 0 & 1/3 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[I_2, 0] = PAQ$$

Hence $P = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -1/3 & 8/3 \\ 0 & 1/3 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}$.

Example 3.40. *If* $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$. *Find two non-singular matrices P and Q such that PAQ = I.*

Solution.

$$[A]_{3\times3} = I_3\, A\, I_3$$

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$
$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to -C_2$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & -3 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 - C_2$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & -3 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_3 = PAQ$$

i.e.
$$P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & -3 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 3.41. *Find the non singular matrices P and Q such that PAQ is in the normal form when*

$$A = \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 4 & 6 & 1 \\ 2 & -3 & 1 & -2 \end{bmatrix}.$$

Solution. $A = I_3 A I_4$

$$\begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 4 & 6 & 1 \\ 2 & -3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 4 & 6 & 1 \\ 3 & 1 & 2 & 1 \\ 2 & -3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 4 & 6 & 1 \\ 0 & -11 & -16 & -2 \\ 0 & -11 & -11 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - 4C_1$
$C_3 \to C_3 - 6C_1$
$C_4 \to C_4 - C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -11 & -16 & -2 \\ 0 & -11 & -11 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -4 & -6 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -11 & -11 & -4 \\ 0 & -11 & -16 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 1 & -3 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -4 & -6 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2\,(-1/11)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 4/11 \\ 0 & -11 & -16 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2/11 & -1/11 \\ 1 & -3 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -4 & -6 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 + 11R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 4/11 \\ 0 & 0 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2/11 & -1/11 \\ 1 & -1 & -1 \end{bmatrix} A \begin{bmatrix} 1 & -4 & -6 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 - C_2,\ C_4 \to C_4 - \frac{4}{11} C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2/11 & -1/11 \\ 1 & -1 & -1 \end{bmatrix} A \begin{bmatrix} 1 & -4 & -2 & 5/11 \\ 0 & 1 & -1 & -4/11 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to \dfrac{-1}{5} R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2/5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2/11 & -1/11 \\ -1/5 & 1/5 & 1/5 \end{bmatrix} A \begin{bmatrix} 1 & -4 & -2 & 5/11 \\ 0 & 1 & -1 & -4/11 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 \to C_4 + C_3\left(\dfrac{2}{5}\right)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2/11 & -1/11 \\ -1/5 & 1/5 & 1/5 \end{bmatrix} A \begin{bmatrix} 1 & -4 & -2 & -19/55 \\ 0 & +1 & -1 & -42/55 \\ 0 & 0 & 1 & 2/5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[I_3, 0] = PAQ$$

Hence $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2/11 & -1/11 \\ -1/5 & 1/5 & 1/5 \end{bmatrix}$ $Q = \begin{bmatrix} 1 & -4 & -2 & -19/55 \\ 0 & 1 & -1 & -42/55 \\ 0 & 0 & 1 & 2/5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Verification:

$$PAQ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2/11 & -1/11 \\ -1/5 & 1/5 & 1/5 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 4 & 6 & 1 \\ 2 & -3 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -4 & -2 & -19/55 \\ 0 & 1 & -1 & -42/55 \\ 0 & 0 & 1 & 2/5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 6 & 1 \\ 0 & 1 & 1 & 4/11 \\ 0 & 0 & 1 & -2/5 \end{bmatrix} \begin{bmatrix} 1 & -4 & -2 & -19/55 \\ 0 & 1 & -1 & -42/11 \\ 0 & 0 & 1 & 2/5 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

EXERCISE 3.6

Find the rank of following matrices:

1. $\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 2 \\ 3 & -1 & 3 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & -2 & 3 & 4 \\ 5 & 4 & 1 & 6 \\ 2 & 3 & -1 & 1 \end{bmatrix}$ **3.** $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$

Find the rank of following matrices after reducing them to normal form:

8. $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

9. $\begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

11. $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$

Find two non singular matrices *P* and *Q* such *PAQ* is in normal form for the matrix *A*, where *A* is

12. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

Answers

1. 2	**2.** 2	**3.** 2	**4.** 2
5. 2	**6.** 2	**7.** 2	**8.** 2
9. 3	**10.** 3	**11.** 3	

12. $P = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -2 & -1 & 1 \end{bmatrix} Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ and $PAQ = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

13. $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/3 & -5/3 \\ 1/2 & -1/3 & 1/6 \end{bmatrix} Q = \begin{bmatrix} 1 & 4/7 & 9/119 & 9/217 \\ 0 & 1/7 & -1/7 & -1/7 \\ 0 & 0 & -1/17 & 0 \\ 0 & 0 & 0 & 1/31 \end{bmatrix}$

14. $PAQ = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ where $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

INTERESTING FACTS

- Rank of a matrix even has its application in the domain of data mining and bioinformatics.
- In social sciences, the user preferences for countries are represented in the form of matrices.
- In biological sciences, the level of gene expression is represented using matrices.
- In medical sector, cancer and its subtypes are discovered from their molecular data using matrices.

VIDEO REFERENCES

Rank of a Matrix

System of Linear Equations-II

System of Linear Equations

USES OF ICT

- https://youtu.be/puq3qyEWMWs

APPLICATIONS TO REAL LIFE

- These are vitally used to build mathematical and computer programming modelling.
- If some unknown data needs to be recovered in matrix form, let's say in cyber security space, and it is known that it has a low rank, then it can be recovered very efficiently and easily.
- The rank even gives an idea about the dimension of the image. For example, the mapping of 3D space into a 2D plane will not have a "***full rank***".

3.8 LINEAR SYSTEM OF EQUATIONS

Suppose in my neighbourhood, there is an eccentric shopkeeper. He is convinced that some Indians eat more wheat than rice and some Indians eat more rice than wheat. So he offers only two standard packets. The first packet, call it N, has 5 kg of wheat and 2 kg of rice, whereas the second packet, call it S, has 2 kg of wheat and 5 kg of rice. Let us invent a shorthand. Whenever we write (m, n), we mean m kg of wheat and n kg of rice. Now if I buy 3 packets of N, it means that I am buying 15 kg of wheat and 6 kg of rice, *i.e.*, $3N = 3(5, 2) = (15, 6)$.

Similarly, 2 packets of S means 4 kg of wheat and 10 kg of rice, *i.e.*, $2S = 2(2, 5) = (4, 10)$.

If I buy one of each of the packets , then I would have bought 7 kg of wheat and 7 kg of rice, that is,

$$N + S = (5, 2) + (2, 5) = (5 + 2, 2 + 5) = (7, 7).$$

Thus I need m packets of N or n packets of S or both, there is no problem. Suppose I need 19 kg of wheat and 16 kg of rice. What shall I do? I need to buy x packets of N and y packets of S so that $x(5, 2) + y(2, 5) = (19, 16)$.

That is, $(5x, 2x) + (2y, 5y) = (19, 16)$ or $(5x + 2y, 2x + 5y) = (19, 16)$. Thus I end up solving a system of linear equations

$$5x + 2y = 19$$
$$2x + 5y = 16.$$

Explanation in terms of *n* equation in *n*-unknowns:

A linear system of n equations in n unknowns $x_1, x_2 \ldots\ldots x_n$ is a set of equations of the form

$$\begin{aligned} a_{11} x_1 + \ldots\ldots\ldots\ldots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + \ldots\ldots\ldots\ldots + a_{2n} x_n &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{n1} x_1 + \ldots\ldots\ldots\ldots + a_{nn} x_n &= b_n \end{aligned} \qquad \ldots(1)$$

where aij $(1 \le i, j \le n)$ are the known coefficients, b_i $(1 \le i \le n)$ are given numbers.

Note: The system is called homogeneous if all the *bi's* are zero.

Otherwise it is non-homogeneous.

In the matrix notation, the system (1) can be written as

$$AX = B$$

Where

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots\ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots \\ a_{n1} & a_{n2} & \ldots\ldots\ldots\ldots & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

The augmented matrix of the system (1) can be written as

$$C = [A : B] = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots\ldots\ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots\ldots\ldots\ldots & a_{2n} & b_2 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots & \ldots \\ a_{n1} & a_{n2} & \ldots\ldots\ldots\ldots & a_{nn} & b_n \end{bmatrix}$$

A solution of (1) is the set of number $x_1 \ldots\ldots\ldots\ldots x_n$ that satisfy all the n equations.

Remark: Augmented Matrix: An augmented matrix from a system of equations is a matrix of numbers in which each row represents the constants from one equation (both the coefficient and the constant on the other side of the equal sign) and each column represents all the coefficients for a single variable.

3.8.1 Types of Linear Equations

A. Non-Homogeneous Equations
B. Homogeneous Equations

3.8.1.1 Non-Homogeneous Systems

i. **Consistent:** A system of equations is said to be consistent, if they have one or more solution *i.e.*

$x + 2y = 4$	$x + 2y = 4$
$3x + 2y = 2$	$3x + 6y = 12$
Unique solution	Infinite solution

ii. Inconsistent: If a system of equation has no solution, it is said to be inconsistent.

$$x + 2y = 4$$
$$3x + 6y = 5$$

Consistency of a System of Linear equations:

$$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2n}x_n = b_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots\dots\dots + a_{mn}x_n = b_m$$

In matrix notation, these of equations are written as:

$$AX = B$$

where $$A = \begin{bmatrix} a_{11} & a_{12} & \dots\dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots\dots & a_{2n} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ a_{m1} & a_{m2} & \dots\dots\dots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Augmented matrix is $$C = [A : B] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & : & b_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & : & b_m \end{bmatrix}$$

a. **Consistent equation:**

If Rank A = Rank [A : B]

i. Unique solution: Rank A = Rank $[A : B] = n$ (no. of unknowns)

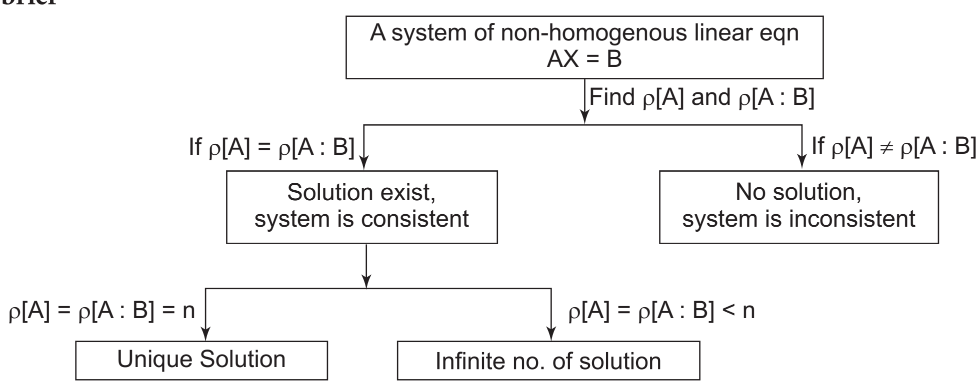
ii. Infinite solution: Rank A = Rank $[A : B] < n$ (no. of unknowns)

b. **Inconsistent equation:**

If Rank $A \neq$ Rank $[A : B]$

Thus we can say that the system of linear equations are either non-homogeneous or Homogeneous. it depends on b_i, then system of non-homogeneous equations are either consistent or inconsistent according to which system has unique solution or infinitely many solution.

In brief



- n = no. of unknowns.

SOME SOLVED EXAMPLES

Example 3.42. *Show that the equations*

$$2x + 6y = -11$$
$$6x + 20y - 6z = -3$$
$$6y - 18z = -1$$

are not consistent.

Solution. The given system of equations in matrix form is

$$\begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix}$$

$$AX = B$$

where $A = \begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix} X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} B = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix}$

Augmented matrix $[A : B]$ is

$$= \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 6 & 20 & -6 & : & -3 \\ 0 & 6 & -18 & : & -1 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$ $$\sim \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 6 & -18 & : & -1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$ $$\sim \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 0 & 0 & : & -91 \end{bmatrix}$$

$$\rho(A) = 2 \text{ and } \rho(A : B) = 3$$

$\therefore$ $$\rho(A) \neq \rho(A : B)$$

Hence, the system of equations are not consistent.

Example 3.43. *Test for consistency and solve*

$$5x + 3y + 7z = 4$$
$$3x + 26y + 2z = 9$$
$$7x + 2y + 10z = 5$$

Solution. The given system of equations in matrix form is

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

$$AX = B$$

where $$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

Augmented matrix $[A : B]$ is

$$\begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$$

$R_1 \to R_1\left(\frac{1}{5}\right)$
$$\sim \begin{bmatrix} 1 & 3/5 & 7/5 & : & 4/5 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 7R_1$
$$\sim \begin{bmatrix} 1 & \frac{3}{5} & \frac{7}{5} & : & \frac{4}{5} \\ 0 & \frac{121}{5} & \frac{-11}{5} & : & \frac{33}{5} \\ 0 & \frac{-11}{5} & \frac{1}{5} & : & \frac{-3}{5} \end{bmatrix}$$

$R_3 \to R_3 + \frac{1}{11}R_2$
$$\sim \begin{bmatrix} 1 & \frac{3}{5} & \frac{7}{5} & : & \frac{4}{5} \\ 0 & \frac{121}{5} & \frac{-11}{5} & : & \frac{33}{5} \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$\rho(A) = 2 \text{ and } \rho[A : B] = 2$$

$\therefore$ $$\rho(A) = \rho[A : B]$$

Hence, the given system of equations are consistent.

But $\rho(A) = \rho[A : B] < n$ (no. of unknowns)

So, its solutions are infinite.

$$x + \frac{3}{5}y + \frac{7}{5}z = \frac{4}{5} \quad \text{...(1)}$$

$$\frac{121}{5}y - \frac{11}{5}z = \frac{33}{5} \quad \Rightarrow \quad 11y - z = 3$$

Let $z = k$, then $11y = 3 + k$

$$y = \frac{3}{11} + \frac{k}{11}$$

Put value of y and z in (1)

$$x + \frac{3}{5}\left(\frac{3}{11} + \frac{k}{11}\right) + \frac{7}{5}k = \frac{4}{5}$$

$$x + \frac{9}{55} + \frac{3k}{55} + \frac{7k}{5} = \frac{4}{5}$$

$$x = \frac{4}{5} - \frac{9}{55} - \left(\frac{3k}{55} + \frac{7k}{5}\right)$$

$$x = \frac{7}{11} - \frac{16}{11}k$$

Example 3.44. *Test for consistency of the following system of equations:*

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 5$$
$$6x_1 + 7x_2 + 8x_3 + 9x_4 = 10$$
$$11x_1 + 12x_2 + 13x_3 + 14x_4 = 15$$
$$16x_1 + 17x_2 + 18x_3 + 19x_4 = 20$$
$$21x_1 + 22x_2 + 23x_3 + 24x_4 = 25$$

Solution. The given system of equations in matrix form is

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \\ 21 & 22 & 23 & 24 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 15 \\ 20 \\ 25 \end{bmatrix}$$

$$AX = B$$

The augmented matrix $[A : B]$ is

$$\begin{bmatrix} 1 & 2 & 3 & 4 & : & 5 \\ 6 & 7 & 8 & 9 & : & 10 \\ 11 & 12 & 13 & 14 & : & 15 \\ 16 & 17 & 18 & 19 & : & 20 \\ 21 & 22 & 23 & 24 & : & 25 \end{bmatrix}$$

$R_2 \to R_2 - 6R_1$
$R_3 \to R_3 - 11R_1$
$R_4 \to R_4 - 16R_1$
$R_5 \to R_5 - 21R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & : & 5 \\ 0 & -5 & -10 & -15 & : & -20 \\ 0 & -10 & -20 & -30 & : & -40 \\ 0 & -15 & -30 & -45 & : & -60 \\ 0 & -20 & -40 & -60 & : & -80 \end{bmatrix} \Rightarrow (-1)^4 \begin{bmatrix} 1 & 2 & 3 & 4 & : & 5 \\ 0 & 5 & 10 & 15 & : & 20 \\ 0 & 10 & 20 & 30 & : & 40 \\ 0 & 15 & 30 & 45 & : & 60 \\ 0 & 20 & 40 & 60 & : & 80 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$
$R_4 \to R_4 - 3R_2$
$R_5 \to R_5 - 4R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & : & 5 \\ 0 & 5 & 10 & 15 & : & 20 \\ 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$\rho(A) = 2 \text{ and } \rho[A : B] = 2$$

$\therefore$ $$\rho(A) = \rho[A : B]$$

Hence, the given system of equations are consistent.

But $$\rho(A) = \rho[A : B] < n$$

So, its solutions are infinite.

Example 3.45. *For what value of k, the system*

$$x + y + z = 1$$
$$2x + y + 4z = k$$
$$4x + y + 10z = k^2 \text{ has a solution.}$$

Solution. The given system of equations in matrix form are

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix}$$

$$AX = B$$

Augmented matrix $[A : B]$

$$\begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 2 & 1 & 4 & : & k \\ 4 & 1 & 10 & : & k^2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & -1 & 2 & : & k-2 \\ 0 & -3 & 6 & : & k^2-4 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & -1 & 2 & : & k-2 \\ 0 & 0 & 0 & : & k^2-3k+2 \end{bmatrix}$$

If the given system has solution, then

$$\rho(A) = \rho[A : B]$$

and $\rho[A : B] = 2$ if $k^2 - 3k + 2 = 0$

$$k^2 - 2k - k + 2 = 0$$
$$(k-2)(k-1) = 0$$
$$k = 2, k = 1.$$

Case I: When $k = 1$, we have

$$x + y + z = 1 \quad ...(1)$$
$$-y + 2z = 1 - 2 = -1 \quad ...(2)$$

Let $$z = \lambda$$

Putting value of $z = \lambda$ in (2) $y = 2\lambda + 1$

Putting the value of y and z in (1)

$$x + (2\lambda + 1) + \lambda = 1$$

$$x + 3\lambda + 1 = 1$$
$$x = -3\lambda$$

Case II: When $k = 2$

$$x + y + z = 1 \quad ...(3)$$
$$-y + 2z = 2 - 2 = 0 \quad ...(4)$$

Let $z = c$

Putting the value of z in (4)

$$-y + 2c = 0$$
$$y = 2c$$

Putting the value of y and z in (3)

$$x + 2c + c = 1$$
$$x = 1 - 3c$$

Example 3.46. *Investigate the values of* λ *and* μ *so that the equations.*

$$2x + 3y + 5z = 9$$
$$7x + 3y - 2z = 8$$
$$2x + 3y + \lambda z = \mu$$

have i. *no solution*

ii. *a unique solution*

iii. *an infinite no. of solution.*

Solution. The given system of equations in matrix form is

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

Augmented matrix is $[A : B]$

$$\begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 7 & 3 & -2 & : & 8 \\ 2 & 3 & \lambda & : & \mu \end{bmatrix}$$

$R_2 \to R_2 - \frac{7}{2}R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 0 & \frac{-15}{2} & \frac{-39}{2} & : & \frac{-47}{2} \\ 0 & 0 & \lambda - 5 & : & \mu - 9 \end{bmatrix}$$

i. No solution: Rank $A \neq$ Rank $(A : B)$

$$\lambda - 5 = 0 \qquad \mu - 9 \neq 0$$
$$\lambda = 5 \qquad \mu \neq 9$$

ii. A unique solution Rank A = Rank $(A : B) = n$

$$\lambda - 5 \neq 0$$
$$\lambda \neq 5, \mu \text{ is arbitrary}$$

iii. An infinite no. of solutions: Rank A = Rank $(A : B) < n$

$$\lambda - 5 = 0 \qquad \mu - 9 = 0$$
$$\lambda = 5 \qquad \mu = 9$$

Example 3.47. *Determine for what value of λ and μ the following equation have*

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

i. *no solution*
ii. *a unique solution*
iii. *infinite number of solution.*

Solution. The given system of equations in matrix form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$AX = B$$

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda - 1 & : & \mu - 6 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda - 3 & : & \mu - 10 \end{bmatrix}$$

i. No solution: $\rho(A) \neq \rho[A : B]$

$$\lambda - 3 = 0 \qquad \mu - 10 \neq 0$$
$$\lambda = 3 \qquad \mu \neq 10$$

ii. A unique solution: $\rho(A) = \rho[A : B] = n$

$$\lambda - 3 \neq 0$$
$$\lambda \neq 3, \mu \text{ is arbitrary.}$$

iii. infinite solution: $\rho(A) = \rho[A : B] < n$

$$\lambda - 3 = 0 \qquad \mu - 10 = 0$$
$$\lambda = 3 \qquad \mu = 10$$

Example 3.48. *Show that the equation*

$$-2x + y + z = a$$
$$x - 2y + z = b$$
$$x + y - 2z = c$$

have no solution unless $a + b + c = 0$. In which case they have infinitely many solution? Find these solutions when $a = 1$, $b = 1$ and $c = -2$.

Solution. The given system of equations in matrix form is

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$AX = B$$

The augmented matrix is $[A : B]$

$$\begin{bmatrix} -2 & 1 & 1 & : & a \\ 1 & -2 & 1 & : & b \\ 1 & 1 & -2 & : & c \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & -2 & 1 & : & b \\ -2 & 1 & 1 & : & a \\ 1 & 1 & -2 & : & c \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$
$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & -2 & 1 & : & b \\ 0 & -3 & 3 & : & a + 2b \\ 0 & 3 & -3 & : & c - b \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & -2 & 1 & : & b \\ 0 & -3 & 3 & : & a + 2b \\ 0 & 0 & 0 & : & a + b + c \end{bmatrix}$$

Case I: If $a + b + c \neq 0$

$$\rho(A) = 2 \text{ and } \rho(A : B) = 3$$

$\therefore \quad \rho(A) \neq \rho(A : B)$

Hence, the system being inconsistent, has no solution.

Case II: If $a + b + c = 0$

$$\rho(A) = 2, \rho(A : B) = 2$$

$\therefore \quad \rho(A) = \rho(A : B)$

Hence, the given system of equations are consistent.

But $\quad \rho(A) = \rho(A : B) < n$

So, its have infinite no. of solution.

Case III: On putting $a = 1, b = 1, c = -2$

$$\begin{bmatrix} 1 & -2 & 1 & : & 1 \\ 0 & -3 & 3 & : & 3 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$x - 2y + z = 1 \quad ...(1)$$

$$-3y + 3z = 3 \quad ...(2)$$

$$-y + z = 1$$

Put $z = k$, $\quad y = k - 1$

Put the value of y and z in (1)

$$x - 2(k-1) + k = 1$$
$$x - 2k + 2 + k = 1$$
$$x - k + 2 = 1$$
$$x = k - 1$$
$$x = k-1, \quad y = k-1, \quad z = k$$

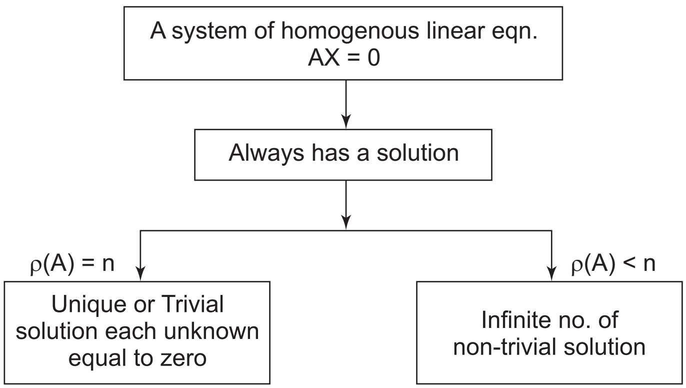
3.8.1.2 Homogeneous Equations

For a system of homogeneous linear equations $AX = 0$.

i. $X = 0$ is always a solution. This solution in which each unknown has the value zero is called the null solution or the Trivial solution. Thus a Homogeneous system is always consistent.
   - A system of Homogeneous linear equations has either the trivial solution or infinite no. of solutions.

ii. If $\rho(A)$ = no. of unknowns the system has only trivial solution.

iii. If $\rho(A)$ < no. of unknowns, the system has infinite no. of non-trivial solutions.



Example 3.49. *Determine the value of b such that the system of Homogeneous equations*

$$2x + y + 2z = 0$$
$$x + y - 3z = 0$$
$$4x + 3y + bz = 0$$

has i. *Trivial solution*

ii. *Non-trivial solution. Find Non-trivial solution using matrix method.*

Solution. i. ***For Trivial solution:*** We know that $x = 0, y = 0\ z = 0$, so b can have any value.

ii. ***For Non-Trivial solution:*** The given system of equation in matrix form is

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & -3 \\ 4 & 3 & b \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = B$$

Augmented matrix is $[A : B]$

$$\begin{bmatrix} 2 & 1 & 2 & : & 0 \\ 1 & 1 & -3 & : & 0 \\ 4 & 3 & b & : & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$
$$\sim \begin{bmatrix} 1 & 1 & -3 & : & 0 \\ 2 & 1 & 2 & : & 0 \\ 4 & 3 & b & : & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 4R_1$
$$\sim \begin{bmatrix} 1 & 1 & -3 & : & 0 \\ 0 & -1 & 8 & : & 0 \\ 0 & -1 & b+12 & : & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$$\sim \begin{bmatrix} 1 & 1 & -3 & : & 0 \\ 0 & -1 & 8 & : & 0 \\ 0 & 0 & b+4 & : & 0 \end{bmatrix}$$

For Non-trivial solution $\rho[A] = \rho[A : B] < n$

$$b + 4 = 0$$
$$b = -4$$

Example 3.50. *Find the value of k such that system of equations*

$$x + ky + 3z = 0$$
$$4x + 3y + kz = 0$$
$$2x + y + 2z = 0$$

has non trivial solution.

Solution. The given system of equation in matrix form is

$$\begin{bmatrix} 1 & k & 3 \\ 4 & 3 & k \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = 0$$

The augmented matrix is $[A : B]$

$$\begin{bmatrix} 1 & k & 3 & : & 0 \\ 4 & 3 & k & : & 0 \\ 2 & 1 & 2 & : & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$
$$\sim \begin{bmatrix} 2 & 1 & 2 & : & 0 \\ 4 & 3 & k & : & 0 \\ 1 & k & 3 & : & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - \frac{1}{2}R_1$
$$\sim \begin{bmatrix} 2 & 1 & 2 & : & 0 \\ 0 & 1 & k-4 & : & 0 \\ 0 & k-\frac{1}{2} & 2 & : & 0 \end{bmatrix}$$

$$R_3 \to R_3 - \left(k - \frac{1}{2}\right)R_2 \qquad \sim \begin{bmatrix} 2 & 1 & 2 & : & 0 \\ 0 & 1 & k-4 & : & 0 \\ 0 & 0 & 2 - \left(k - \frac{1}{2}\right)(k-4) & : & 0 \end{bmatrix}$$

For Non-trivial solution $\rho[A] = \rho[A : B] < n$

$$2 - \left(k - \frac{1}{2}\right)(k-4) = 0$$

$$2 - k^2 + 4k + \frac{k}{2} - 2 = 0$$

$$-k^2 + \frac{9}{2}k = 0$$

$$k\left(-k + \frac{9}{2}\right) = 0$$

$$k = 0, k = \frac{9}{2}.$$

EXERCISE 3.7

Check the consistency of the following system of equations. Also find the solution set:

1.
$$x + 2y - z = 3$$
$$3x - y + 2z = 1$$
$$2x - 2y + 3z = 2$$
$$x - y + z = -1$$

2.
$$x + 3y - z = 4$$
$$2x + y + z = 7$$
$$2x - 4y + 4z = 6$$
$$3x + 4y = 1$$

3.
$$x_1 + 2x_2 - x_3 = 6$$
$$3x_1 - x_2 + 2x_3 = 3$$
$$4x_1 - 3x_2 + x_3 = 9$$

4.
$$x - 4y - 3z = -16$$
$$2x + 7y + 12z = 48$$
$$4x - y + 6z = 16$$
$$5x - 5y + 3z = 0$$

5. Discuss the consistency of the equation

$$x + 2y + 3z + 4t = 0$$
$$2x + 3y + 4z - 1 = 0$$
$$3x + 4y + t = 2$$
$4x + z + 2t = 3$ Find the solution set if consistent.

6. For what value of λ will the equations

$$3x - y + \lambda z = 0$$
$$2x + y + z = 2$$
$x + 2y - \lambda z = -1$ Fail to have a unique solution.

Will the equations have any solution for this value of λ?

7. Use the test of rank to show that the following system of equations is inconsistent:

$$2x - y + z = 4$$
$$3x - y + z = 6$$
$$4x - y + 2z = 7$$
$$-x + y - z = 9$$

8. Show that the following equations are consistent and solve them:

$$x + 2y - 5z = -9$$
$$3x - y + 2z = 5$$
$$2x + 3y - z = 3$$
$$4x - 5y + z = -3$$

9. Solve the system of equations:

$$\lambda x + 2y - 2z = 1$$
$$4x + 2\lambda y - z = 2$$
$$6x + 6y + \lambda z = 3 \text{ considering specially the case when } \lambda = 2$$

10. For what value of a and b the equation

$$x + y + 5z = 0$$
$$x + 2y + 3az = b$$
$$x + 3y + az = 1 \text{ have}$$

i. No solution ii. unique solution iii. infinitely many solutions

Solve the following system of equations:

11.
$$x - y + z = 0$$
$$-3x + y - 4z = 0$$
$$7x - 3y - 9z = 0$$
$$4x - 2y + 5z = 0$$

12.
$$x + 3y - 2z = 0$$
$$2x - y + 4z = 0$$
$$x - 11y + 14z = 0$$

13.
$$2w + 3x - y - z = 0$$
$$4w - 6x - 2y + 2z = 0$$
$$-6w + 12x + 3y - 4z = 0$$
$$8w - 24x - 4y + 8z = 0$$

14.
$$3x + 4y - z - 6w = 0$$
$$2x + 3y + 2z - 3w = 0$$
$$2x + y - 14z - 9w = 0$$
$$x + 3y + 13z + 3w = 0$$

15. Find the value of k such that following system of equations has a non-trivial solution:

$$(3k - 8)x + 3y + 3z = 0$$
$$3x + (3k - 8)y - 3z = 0$$
$$3x + 3y + (3k - 8)z = 0$$

16. Show that the only real value of λ for which the equations:

$$x + 2y + 3z = \lambda x$$
$$3x + y + 2z = \lambda y$$
$$2x + 3y + z = \lambda z \text{ have a non-zero solution 6.}$$

Answers

1. $-1, 4, 4$
2. Not consistent
3. $-1, 4, 4$
4. $\frac{17}{5} - \frac{4}{5}k, \frac{1}{5} + \frac{3}{5}k, k$

5. Consistent, $\left(\frac{9}{11}, \frac{-1}{11}, \frac{-1}{11}, \frac{-1}{11}\right)$ **6.** Inconsistent, $\lambda = -\frac{7}{2}$ **8.** $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$

9. $x = \frac{1}{2} - k, y = k, z = 0$ **10.** i. $a = 1, b \neq \frac{1}{2}$ ii. $a \neq 1, b \in R$ iii. $a = 1, b = \frac{1}{2}$

11. $x = y = z = 0$ **12.** $x = \frac{-10}{7}k, y = \frac{8}{7}k, z = k$ **13.** $x = \frac{1}{3}k_1,\ y = k_2, z = k_1,\ w = \frac{1}{2}k_2$

14. $x = 11k_1 + 6k_2, y = -8k_1 - 3k_2, z = k_1, w = k_2$ **15.** Non trivial solution if $k = \frac{11}{3}$ or $k = \frac{2}{3}$

3.9 SOLUTION OF SYSTEM OF LINEAR EQUATIONS BY DETERMINANTS

3.9.1 Cramer's Rule

This method is given by Swiss mathematician Gabriel Cramer. We will explain this method by considering the following system of equations:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Let $$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

then $$xD = \begin{vmatrix} xa_1 & b_1 & c_1 \\ xa_2 & b_2 & c_2 \\ xa_3 & b_3 & c_3 \end{vmatrix}$$

Operating $C_1 \to C_1 + yC_2 + zC_3$, we have

$$xD = \begin{vmatrix} xa_1 + yb_1 + zc_1 & b_1 & c_1 \\ xa_2 + yb_2 + zc_2 & b_2 & c_2 \\ xa_3 + yb_3 + zc_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$= D_1 \text{ (say)}$$

then $$x = \frac{D_1}{D}$$

$$= \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Similarly, $y = \dfrac{D_2}{D}$ amd $z = \dfrac{D_3}{D}$

where $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ and $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Thus, $x = \dfrac{D_1}{D}$, $y = \dfrac{D_2}{D}$, $z = \dfrac{D_3}{D}$ are the values of unknowns given in the system of equations.

SOME SOLVED EXAMPLES

Example 3.51. *Solve the given system of equations using Cramer's Rule*

$$x + y + z = 1,\ 3x + 5y + 6z = 4,\ 9x + 2y - 36z = 17.$$

Solution. Given

$$x + y + z = 1$$
$$3x + 5y + 6z = 4$$
$$9x + 2y - 36z = 17$$

Here,

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ 9 & 2 & -36 \end{vmatrix}$$

$$= 1(-180 - 12) - 1(-108 - 54) + 1(6 - 45)$$
$$= -192 + 162 - 39 = -69$$

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 5 & 6 \\ 17 & 2 & -36 \end{vmatrix}$$

$$= 1(-180 - 12) - 1(-144 -102) + 1(8 - 85)$$
$$= -192 + 246 - 77 = -23$$

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 4 & 6 \\ 9 & 17 & -36 \end{vmatrix}$$

$$= 1(-144 - 102) - 1(-108 - 54) + 1(51 - 36)$$
$$= -246 + 162 + 15 = -69$$

$$D_3 = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 9 & 2 & 17 \end{vmatrix}$$

$$= 1(85 - 8) - 1(51 - 36) + 1(6 - 45)$$
$$= 77 - 15 - 39 = 23$$

By Cramer's Rule, we have

$$x = \frac{D_1}{D}, \; y = \frac{D_2}{D}, \; z = \frac{D_3}{D}$$

$$x = \frac{-23}{-69} = \frac{1}{3}, \; y = \frac{-69}{-69} = 1, \; z = \frac{23}{-69} = -\frac{1}{3} \quad \textbf{Answer}$$

Example 3.52. *Solve the given system of equations using Cramer's Rule*

$3x + y + 2z = 3$, $2x - 3y - z = -3$, $x + 2y + z = 4$.

Solution. Given

$$3x + y + 2z = 3$$
$$2x - 3y - z = -3$$
$$x + 2y + z = 4$$

Here,

$$D = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= 3(-3 + 2) - 1(2 + 1) + 2(4 + 3)$$
$$= 3(-1) - 1(3) + 2(7)$$
$$= -3 - 3 + 14 = 8$$

$\therefore$

$$D_1 = \begin{vmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{vmatrix}$$

$$= 3(-3 + 2) - 1(-3 + 4) + 2(-6 + 12)$$
$$= 3(-1) - 1(1) + 2(6)$$
$$= -3 - 1 + 12 = 8$$

$$D_2 = \begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{vmatrix}$$

$$= 3(-3 + 4) - 3(2 + 1) + 2(8 + 3)$$
$$= 3(1) - 3(3) + 2(11)$$
$$= 3 - 9 + 22 = 16$$

$$D_3 = \begin{vmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix}$$

$$= 3(-12 + 6) - 1(8 + 3) + 3(4 + 3)$$
$$= 3(-6) - (1)(11) + 3(7)$$
$$= -18 - 11 + 21 = -8$$

Thus, $x = \frac{D_1}{D} = \frac{8}{8} = 1, \quad y = \frac{D_2}{D} = \frac{16}{8} = 2, \quad z = \frac{D_3}{D} = \frac{-8}{8} = -1$ **Answer**

EXERCISE 3.8

Apply Cramer's Rule to solve the following equations:

1. $x + 3y + 6z = 2, 3x - y + 4z = 9, x - 4y + 2z = 7.$
2. $x + y + z = -1, x + 2y + 4z = -5, 6x + 4y + 2z = 0.$
3. $x - 4y - z = 11, 2x - 5y + 2z = 39, -3x + 2y + z = 1.$
4. $x + 2y + 3z = 6, 2x + 4y + z = 7, 3x + 2y + 9z = 14.$

Answers

1. $x = 2, y = -1, z = 1/2$
2. $x = 1, y = -1, z = -1$
3. $x = -1, y = -5, z = 8$
4. $x = y = z = 1$

3.9.2 Gauss Elimination Method (To Solve System of Linear Equations)

Consider the system of equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad \text{...(1)}$$

The system in matrix form is $AX = B$, where

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, B = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \text{and } X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Consider the augmented matrix $[A : B]$

$$[A : B] = \begin{pmatrix} a_1 & b_1 & c_1 & : & d_1 \\ a_2 & b_2 & c_2 & : & d_2 \\ a_3 & b_3 & c_3 & : & d_3 \end{pmatrix} \quad \text{...(2)}$$

Now Eq. (2) can be reduced to an upper triangular matrix Let $a_1 \neq 0$, then

$$R_2 \to R_2 - \frac{a_2}{a_1}R_1, R_3 \to R_3 - \frac{a_3}{a_1}R_1 = \begin{pmatrix} a_1 & b_1 & c_1 & : & d_1 \\ 0 & b_2' & c_2' & : & d_2' \\ 0 & b_3' & c_3' & : & d_3' \end{pmatrix} \quad \text{...(3)}$$

Here a_1 is called first pivot.

Now, take b_2' as the pivot ($b_2' \neq 0$), then

$$R_3 \to R_3 - \frac{b_3'}{b_2'}R_2 \sim \begin{pmatrix} a_1 & b_1 & c_1 & : & d_1 \\ 0 & b_2' & c_2' & : & d_2' \\ 0 & 0 & c_3'' & : & d_3'' \end{pmatrix} \quad \text{...(4)}$$

Now take $c_3'' \neq 0$ as the pivot from Eq. (4), the given system of linear equation is equivalent to

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \end{aligned}$$

$$c''_3 z = d''_3$$

or $$z = \frac{d''_3}{c''_3}$$

By back substitution $$y = \frac{(c_3''d_2' - c_2'd_3'')}{b_2'c_3''}$$

and $$x = \frac{1}{a_1 b'_2 c''_3}(d_1 b'_2 c''_3 + b_1 c'_2 d''_3 - b'_2 c_1 d''_3 - b_1 c''_3 d'_2)$$

Notes:

1. This method fails if anyone of the pivots a_1, b'_2, c''_3 becomes zero. In such cases by interchanging the rows, we can get the nonzero pivots.
2. **Partial pivoting:** From the first column of Eq. (2), select the component with the largest absolute value. This is called pivot. Then at the second stage, i.e. from the second column of Eq. (3), select once again the component with largest absolute value as the pivot. Continue this process. This procedure is called partial pivoting.
3. **Complete pivoting:** If one is not interested in the elimination of x, y, z in a particular order, then choose at each stage numerically the largest coefficient of the entire coefficient matrix. This requires an interchange of equations and also an interchange of positions of the variables.

SOME SOLVED EXAMPLES

Example 3.53. *Solve the system of equations $3x + y - z = 3$, $2x - 8y + z = -5$, $x - 2y + 9z = 8$, using Gauss elimination method.*

Solution. The given method is equivalent to $\begin{pmatrix} 3 & 1 & -1 \\ 2 & -8 & 1 \\ 1 & -2 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 8 \end{pmatrix}$

The augmented matrix is

$$[A : B] = \begin{pmatrix} 3 & 1 & -1 & : & 3 \\ 2 & -8 & 1 & : & -5 \\ 1 & -2 & 9 & : & 8 \end{pmatrix}$$

$$R_2 \to R_2 - \frac{2}{3}R_1, R_3 \to R_3 - \frac{1}{3}R_1 \sim \begin{pmatrix} 3 & 1 & -1 & : & 3 \\ 0 & \frac{-26}{3} & \frac{5}{3} & : & -7 \\ 0 & \frac{-7}{3} & \frac{28}{3} & : & 7 \end{pmatrix}$$

Choosing $\frac{-26}{3}$ as the pivot from the 2nd column, we have

$$R_3 \to R_3 - \frac{7}{26}R_3 \sim \begin{pmatrix} 3 & 1 & -1 & : & 3 \\ 0 & \frac{-26}{3} & \frac{5}{3} & : & -7 \\ 0 & 0 & \frac{2079}{234} & : & \frac{231}{26} \end{pmatrix}$$

or
$$3x + y - z = 3$$
$$\frac{-26}{3}y + \frac{5}{3}z = -7$$
$$\frac{2079}{234}z = \frac{231}{26} \text{ or } z = \frac{231 \times 234}{26 \times 2079} = 1$$

Now by back substitution, $z = 1$

$$-26y = -26 \quad \text{or} \quad y = 1$$

and
$$3x = 3x \quad \text{or} \quad x = 1$$
$$x = 1, y = 1, z = 1$$

Example 3.54. *Using Gauss elimination method, solve the system of equations*

$$3.15x - 1.96y + 3.85z = 12.95$$
$$2.13x + 5.12y - 2.89z = -8.61$$
$$5.92x + 3.05y + 2.15z = 6.88$$

Solution. The given system is equivalent to

$$\begin{pmatrix} 3.15 & -1.96 & 3.85 \\ 2.13 & 5.12 & -2.89 \\ 5.92 & 3.05 & 2.15 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12.95 \\ -8.61 \\ 6.88 \end{pmatrix}$$

$$AX = B$$

$$[A : B] = \begin{pmatrix} 3.15 & -1.96 & 3.85 & : & 12.95 \\ 2.13 & 5.12 & -2.89 & : & -8.61 \\ 5.92 & 3.05 & 2.15 & : & 6.88 \end{pmatrix}$$

Choosing 3.15 as pivot

$$R_2 \to R_2 - \frac{2.13}{3.15}R_1, R_3 \to R_3 - \frac{5.92}{3.15}R_1 \sim \begin{pmatrix} 3.15 & -1.96 & 3.85 & : & 12.95 \\ 0 & 6.4453 & -5.4933 & : & -17.3667 \\ 0 & 6.7335 & -5.0855 & : & -17.4578 \end{pmatrix}$$

Choosing 6.4453 as pivot

$$R_3 \to R_3 - \frac{6.7335}{6.4453}R_2 \sim \begin{pmatrix} 3.15 & -1.96 & 3.85 & : & 12.95 \\ 0 & 6.4453 & -5.4933 & : & -17.3667 \\ 0 & 0 & 0.6534 & : & 0.6854 \end{pmatrix}$$

$$3.15x - 1.96y + 3.85z = 12.95$$
$$6.4453y - 5.4933z = -17.3667$$
$$0.6534z = 0.6854$$

By back substitution $z = \dfrac{0.6854}{0.6534} = 1.04897459,\ y = \dfrac{5.4933z - 17.3667}{6.4453} = -1.80043875$

and $x = \dfrac{1.96y - 3.85z + 12.95}{3.15} = 1.70875806$

Example 3.55. *Solve the system of equations*

$$x_1 + x_2 + x_3 + x_4 = 2$$
$$x_1 + x_2 + 3x_3 - 2x_4 = -6$$
$$2x_1 + 3x_2 - x_3 + 2x_4 = 7$$
$$x_1 + 2x_2 + x_3 - x_4 = -2.$$

by Gauss elimination method.

Solution. $$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & -2 \\ 2 & 3 & -1 & 2 \\ 1 & 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ 7 \\ -2 \end{pmatrix}$$

$$AX = B$$

$$[A : B] = \begin{pmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 1 & 1 & 3 & -2 & : & -6 \\ 2 & 3 & -1 & 2 & : & 7 \\ 1 & 2 & 1 & -1 & : & -2 \end{pmatrix}$$

$$\begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \end{matrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 0 & 0 & 2 & -3 & : & -8 \\ 0 & 1 & -3 & 0 & : & 3 \\ 0 & 1 & 0 & -2 & : & -4 \end{pmatrix}$$

Since the element in second row, second column is zero, interchange second and third row to get pivot element 1, *i.e.*

$$R_2 \leftrightarrow R_3 \sim \begin{pmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 0 & 1 & -3 & 0 & : & 3 \\ 0 & 0 & 2 & -3 & : & -8 \\ 0 & 1 & 0 & -2 & : & -4 \end{pmatrix}$$

$$R_4 \to R_4 - R_2 \sim \begin{pmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 0 & 1 & -3 & 0 & : & 3 \\ 0 & 0 & 2 & -3 & : & -8 \\ 0 & 0 & 3 & -2 & : & -7 \end{pmatrix}$$

Now the pivot is 2, therefore

$$R_4 \to R_4 - \frac{3}{2}R_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 0 & 1 & -3 & 0 & : & 3 \\ 0 & 0 & 2 & -3 & : & -8 \\ 0 & 0 & 0 & \frac{5}{2} & : & 5 \end{pmatrix}$$

$$x_1 + x_2 + x_3 + x_4 = 2$$
$$x_2 - 3x_3 = 3$$
$$2x_3 - 3x_4 = -8$$
$$\frac{5}{2}x_4 = 5$$
$$\Rightarrow \quad x_4 = 2$$
$$\Rightarrow \quad x_3 = \frac{1}{2}(-8 + 3x_4) = \frac{1}{2}(-8 + 6) = -1$$

Now by back substitution

$$x_2 = 3 + 3x_3 = 3 - 3 = 0$$
$$x_1 = 2 - x_2 - x_3 - x_4 = 2 - 0 - (-1) - 2 = 1$$
$$x_1 = 1, x_2 = 0, x_3 = -1, x_4 = 2.$$

EXERCISE 3.9

Solve the system of equations by Gauss elimination method:

1.
$$x + 2y + z = 3$$
$$2x + 3y + 3z = 10$$
$$3x - y + 2z = 13$$

2.
$$2x + 3y - z = 5$$
$$4x + 4y - 3z = 3$$
$$2x - 3y + 2z = 2$$

3.
$$5x_1 + x_2 + x_3 + x_4 = 4$$
$$x_1 + 7x_2 + x_3 + x_4 = 12$$
$$x_1 + x_2 + 6x_3 + x_4 = -5$$
$$x_1 + x_2 + x_3 + 4x_4 = -6$$

Answers

1. $x = 2, y = -1, z = 3$ **2.** $x = 1, y = 2, z = 3$ **3.** $x_1 = 1, x_2 = 2, x_3 = -1, x_4 = -2$

3.9.3 Gauss-Jordan Method (To Solve System of Linear Equations)

This method is a modified form of Gauss elimination method. The coefficient matrix A of $AX = B$ is reduced to a diagonal matrix or unit matrix by making all the elements above and below the principal diagonal of A as zero. The time of back substitution is saved here, even though it involves additional computations.

SOME SOLVED EXAMPLES

Example 3.56. *Solve the equations*

$$10x + y + z = 12$$
$$2x + 10y + z = 13$$
$$x + y + 5z = 7$$

by Gauss-Jordan method.

Solution. The given system in matrix form is

$$\begin{pmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 13 \\ 7 \end{pmatrix}$$

$$AX = B$$

$$[A : B] = \begin{pmatrix} 10 & 1 & 1 & : & 12 \\ 2 & 10 & 1 & : & 13 \\ 1 & 1 & 5 & : & 7 \end{pmatrix}$$

$$R_1 \to R_1 - 9R_3 \sim \begin{pmatrix} 1 & -8 & -44 & : & -51 \\ 2 & 10 & 1 & : & 13 \\ 1 & 1 & 5 & : & 7 \end{pmatrix}$$

$$\begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \sim \begin{pmatrix} 1 & -8 & -44 & : & -51 \\ 0 & 26 & 89 & : & 115 \\ 0 & 9 & 49 & : & 58 \end{pmatrix}$$

$$R_2 \to 3R_3 - R_2 \sim \begin{pmatrix} 1 & -8 & -44 & : & -51 \\ 0 & 1 & 58 & : & 59 \\ 0 & 9 & 49 & : & 58 \end{pmatrix}$$

$$\begin{matrix} R_1 \to R_1 + 8R_2 \\ R_2 \to R_2 - 9R_2 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 420 & : & 421 \\ 0 & 1 & 58 & : & 59 \\ 0 & 0 & -473 & : & -473 \end{pmatrix}$$

$$R_3 \to -\frac{1}{473}R_3 \sim \begin{pmatrix} 1 & 0 & 420 & : & 421 \\ 0 & 1 & 58 & : & 59 \\ 0 & 0 & 1 & : & 1 \end{pmatrix}$$

$$\begin{matrix} R_1 \to R_1 - 420R_3 \\ R_2 \to R_2 - 58R_3 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & 1 \end{pmatrix}$$

The system $AX = B$ reduces to the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad i.e. \quad x = 1 = y = z.$$

Example 3.57. *Solve the equations*

$10x_1 + x_2 + x_3 = 12$, $x_1 + 10x_2 - x_3 = 10$ and $x_1 - 2x_2 + 10x_3 = 9$ by Gauss-Jordan method.

Solution. The given system in matrix form is

$$\begin{pmatrix} 10 & 1 & 1 \\ 1 & 10 & -1 \\ 1 & -2 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \\ 9 \end{pmatrix}$$

$$AX = B$$

$$[A : B] = \begin{pmatrix} 10 & 1 & 1 & : & 12 \\ 1 & 10 & -1 & : & 10 \\ 1 & -2 & 10 & : & 9 \end{pmatrix}$$

$$R_1 \to R_1 - 9R_2 \sim \begin{pmatrix} 1 & -89 & 10 & : & -78 \\ 1 & 10 & -1 & : & 10 \\ 1 & -2 & 10 & : & 9 \end{pmatrix}$$

$$\begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \sim \begin{pmatrix} 1 & -89 & 10 & : & -78 \\ 0 & 99 & -11 & : & 88 \\ 0 & 87 & 0 & : & 87 \end{pmatrix}$$

$$\begin{matrix} R_2 \to \dfrac{R_2}{11} \\ R_3 \to \dfrac{R_3}{87} \end{matrix} \sim \begin{pmatrix} 1 & -89 & 10 & : & -78 \\ 0 & 9 & -1 & : & 8 \\ 0 & 1 & 0 & ; & 1 \end{pmatrix}$$

$$R_2 \to R_2 - 8R_3 \sim \begin{pmatrix} 1 & -89 & 10 & : & -78 \\ 0 & 1 & -1 & : & 0 \\ 0 & 1 & 0 & ; & 1 \end{pmatrix}$$

$$\begin{matrix} R_1 \to R_1 + 89R_2 \\ R_3 \to R_3 - R_2 \end{matrix} \sim \begin{pmatrix} 1 & 0 & -79 & : & -78 \\ 0 & 1 & -1 & : & 0 \\ 0 & 0 & 1 & : & 1 \end{pmatrix}$$

$$\begin{matrix} R_1 \to R_1 + 79R_3 \\ R_2 \to R_2 - R_3 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & 1 \end{pmatrix}$$

The system $AX = B$ reduces to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\therefore \quad x_1 = x_2 = x_3 = 1.$

Example 3.58. *Solve the system of equations*

$$5x - y - 2z = 142$$
$$x - 3y - z = -30$$
$$2x - y - 3z = -5$$

by Gauss-Jordan method.

Solution. Now $[A : B] \sim \begin{pmatrix} 5 & -1 & -2 & : & 142 \\ 1 & -3 & -1 & : & -30 \\ 2 & -1 & -3 & : & -5 \end{pmatrix}$

$$R_1 \leftrightarrow R_2 \sim \begin{pmatrix} 1 & -3 & -1 & : & -30 \\ 5 & -1 & -2 & : & 142 \\ 2 & -1 & -3 & : & -5 \end{pmatrix}$$

$$\begin{matrix} R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} \sim \begin{pmatrix} 1 & -3 & -1 & : & -30 \\ 0 & 14 & 3 & : & 292 \\ 0 & 5 & -1 & : & 55 \end{pmatrix}$$

$$R_2 \to \frac{R_2}{14} \sim \begin{pmatrix} 1 & -3 & -1 & : & -30 \\ 0 & 1 & \frac{3}{14} & : & \frac{146}{7} \\ 0 & 5 & -1 & : & 55 \end{pmatrix}$$

$$\begin{matrix} R_1 \to R_1 + 3R_2 \\ R_3 \to R_3 - 5R_2 \end{matrix} \sim \begin{pmatrix} 1 & 0 & -\frac{5}{14} & : & \frac{228}{7} \\ 0 & 1 & \frac{3}{14} & : & \frac{146}{7} \\ 0 & 0 & -\frac{29}{14} & : & -\frac{345}{7} \end{pmatrix}$$

$$R_3 \to \left(-\frac{14}{29}\right)R_3 \sim \begin{pmatrix} 1 & 0 & -\frac{5}{14} & : & \frac{228}{7} \\ 0 & 1 & \frac{3}{14} & : & \frac{146}{7} \\ 0 & 0 & 1 & : & \frac{690}{29} \end{pmatrix}$$

$$\begin{matrix} R_1 \to R_1 + \frac{5}{14}R_3 \\ R_2 \to R_2 - \frac{3}{14}R_3 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 & : & \frac{8337}{203} \\ 0 & 1 & 0 & : & \frac{3199}{203} \\ 0 & 0 & 1 & : & \frac{690}{29} \end{pmatrix}$$

$\therefore$ $$x = \frac{8337}{203}, y = \frac{3199}{203}, z = \frac{690}{29}.$$

Example 3.59. *Solve the system of equations*

$$x+\frac{y}{2}+\frac{z}{3}=1$$

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=0$$

$$\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=0$$

by Gauss-Jordan method.

Solution.

$$6x + 3y + 2z = 6$$
$$6x + 4y + 3z = 0$$
$$20x + 15y + 12z = 0$$

$$[A:B]=\begin{pmatrix}6 & 3 & 2 & : & 6\\ 6 & 4 & 3 & : & 0\\ 20 & 15 & 12 & : & 0\end{pmatrix}$$

$$R_1\to\frac{R_1}{6}\sim\begin{pmatrix}1 & \frac{1}{2} & \frac{1}{3} & : & 1\\ 6 & 4 & 3 & : & 0\\ 20 & 15 & 12 & : & 0\end{pmatrix}$$

$$\begin{matrix}R_2\to R_2-6R_1\\ R_3\to R_3-20R_1\end{matrix}\sim\begin{pmatrix}1 & \frac{1}{2} & \frac{1}{3} & : & 1\\ 0 & 1 & 1 & : & -6\\ 0 & 5 & \frac{16}{3} & : & -20\end{pmatrix}$$

$$\begin{matrix}R_1\to R_1-\frac{1}{2}R_2\\ R_3\to R_3-5R_2\end{matrix}\sim\begin{pmatrix}1 & 0 & -\frac{1}{6} & : & 4\\ 0 & 1 & 1 & : & -6\\ 0 & 0 & \frac{1}{3} & : & 10\end{pmatrix}$$

$$R_3\to 3R_3\sim\begin{pmatrix}1 & 0 & -\frac{1}{6} & : & 4\\ 0 & 1 & 1 & : & -6\\ 0 & 0 & 1 & : & 30\end{pmatrix}$$

$$\begin{matrix}R_1\to R_1+\frac{1}{6}R_3\\ R_2\to R_2-R_3\end{matrix}\sim\begin{pmatrix}1 & 0 & 0 & : & 9\\ 0 & 1 & 0 & : & -36\\ 0 & 0 & 1 & : & 30\end{pmatrix}$$

$\therefore\quad x = 9, y = -36, z = 30.$

EXERCISE 3.10

Solve the system of equations by Gauss-Jordan method:

1.
$$x + y + z + w = 2$$
$$2x - y + 2z - w = -5$$
$$3x + 2y + 3z + 4w = 7$$
$$x - 2y - 3z + 2w = 5$$

2.
$$10x + y + z = 12$$
$$2x + 10y + z = 13$$
$$x + y + 5z = 7$$

3.
$$3x + 4y + 5z = 18$$
$$2x - y + 8z = 13$$
$$5x - 2y + 7z = 20$$

4.
$$x + 2y + z - w = -2$$
$$2x + 3y - z + 2w = 7$$
$$x + y + 3z - 2w = -6$$
$$x + y + z + w = 2$$

5.
$$10x + y + z = 18.141$$
$$x + 10y + z = 28.140$$
$$x + y + 10z = 38.139$$

Answers

1. $x = 0, y = 1, z = -1, w = 2$

2. $x = 1, y = 1, z = 1$

3. $x = 3, y = 1, z = 1$

4. $x = 1, y = 0, z = -1, w = 2$

5. $x = 1.234, y = 2.348, z = 3.455$

3.9.4 Gauss Elimination Method for Finding the Inverse of a Matrix

Let A be a non-singular square matrix of order three. Then the inverse of A is a matrix X which satisfy the equation $AX = I$, where I is the unit matrix of order three. Now, we have to find the elements of the inverse matrix X.

Let
$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

and
$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

The equation becomes $AX = I$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This matrix is equivalent to three equations, which are equivalent to three system of equations

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{...(1)}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad ...(2)$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad ...(3)$$

The systems (1), (2) and (3) of Eqs. (1)–(3) can be solved by Gauss-elimination procedure. The solution set of each system of Eqs. (1), (2) and (3) will be the corresponding column of the inverse matrix X.

Since the coefficient matrix is same in all the Eqs. (1), (2) and (3), all can be simultaneously solved by forming a definite system

$$[A/I] = \left(\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right)$$

SOME SOLVED EXAMPLES

Example 3.60. *By Gauss elimination Method, find the inverse of* $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 3 & -1 & -4 \end{pmatrix}$.

Solution. The augmented system $[A/I]$ is

$$[A/I] \sim \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 3 & -1 & -4 & 0 & 0 & 1 \end{array}\right)$$

Since the element $a_{11} = 0$, we will interchange the first and second row, the reduced system is

$$[A/I] \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 3 & -1 & -4 & 0 & 0 & 1 \end{array}\right)$$

we get $\quad R_3 \to R_3 + (-3)R_1 \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -7 & -4 & 0 & -3 & 1 \end{array}\right)$

$$R_3 \to R_3 + 7R_2 \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 7 & -3 & 1 \end{array}\right)$$

Thus
$$\left.\begin{aligned} x_{11} + 2x_{21} &= 0 \\ x_{21} + x_{31} &= 1 \\ 3x_{31} &= 7 \end{aligned}\right\} \Rightarrow \begin{aligned} x_{31} &= \frac{7}{3} \\ x_{21} &= -\frac{4}{3} \\ x_{11} &= \frac{8}{3} \end{aligned}$$

$$\left.\begin{aligned} x_{12} + 2x_{22} &= 1 \\ x_{22} + x_{32} &= 0 \\ 3x_{32} &= -3 \end{aligned}\right\} \Rightarrow \begin{aligned} x_{32} &= -1 \\ x_{22} &= 1 \\ x_{12} &= -1 \end{aligned}$$

$$\left.\begin{aligned} x_{13} + 2x_{23} &= 0 \\ x_{23} + x_{33} &= 0 \\ 3x_{33} &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} x_{33} &= \frac{1}{3} \\ x_{23} &= -\frac{1}{3} \\ x_{13} &= \frac{2}{3} \end{aligned}$$

Hence $$A^{-1} = \begin{pmatrix} \frac{8}{3} & -1 & \frac{2}{3} \\ -\frac{4}{3} & 1 & -\frac{1}{3} \\ \frac{7}{3} & -1 & \frac{1}{3} \end{pmatrix}$$

Example 3.61. *Find by Gauss elimination method, the inverse of* $A = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$.

Solution. $$[A/I] = \left(\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ -15 & 6 & -5 & 0 & 1 & 0 \\ 5 & -2 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$R_2 \to R_2 + 5R_1, R_3 \to R_3 - \left(\frac{5}{3}\right)R_1 \sim \left(\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & \frac{-1}{3} & \frac{1}{3} & \frac{-5}{3} & 0 & 1 \end{array}\right)$$

$$R_3 \to R_3 + \left(\frac{1}{3}\right)R_2 \sim \left(\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 1 \end{array}\right)$$

Now the system is equivalent to three systems.

$$\left(\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & \frac{1}{3} & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 3 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \end{array}\right)$$

and
$$\left(\begin{array}{ccc|c} 3 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 1 \end{array}\right)$$

$$\left.\begin{array}{r} 3x_{11} - x_{21} + x_{31} = 1 \\ x_{21} = 5 \\ \frac{1}{3}x_{31} = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x_{31} = 0 \\ x_{21} = 5 \\ x_{11} = 2 \end{array}$$

$$\left.\begin{array}{r} 3x_{12} - x_{22} + x_{32} = 0 \\ x_{22} = 1 \\ \frac{1}{3}x_{32} = \frac{1}{3} \end{array}\right\} \Rightarrow \begin{array}{l} x_{32} = 1 \\ x_{22} = 1 \\ x_{12} = 0 \end{array}$$

$$\left.\begin{array}{r} 3x_{13} - x_{23} + x_{33} = 0 \\ x_{23} = 0 \\ \frac{1}{3}x_{33} = 1 \end{array}\right\} \Rightarrow \begin{array}{l} x_{33} = 3 \\ x_{23} = 0 \\ x_{13} = -1 \end{array}$$

$$A^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

EXERCISE 3.11

1. Find the inverse of the following matrices by Gauss elimination method:

i. $\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ ii. $\begin{pmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{pmatrix}$ iii. $\begin{pmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{pmatrix}$

Answers

1. i. $\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$ ii. $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{pmatrix}$ iii. $\frac{1}{8}\begin{pmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{pmatrix}$

3.9.5 Gauss-Jordan Method for Finding the Inverse of a Matrix

Let A be square matrix of order three and $|A| \neq 0$. Then the inverse of A is a matrix X which satisfies the equation $AX = I$, where I is the unit matrix of order three. Now, we have to find the elements of the inverse matrix X.

Let $$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

and $$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

Then, $$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This equation is equivalent to three equations, which are equivalent to three systems of equations. Solve each system by Gauss-Jordan method. The solution set of each system will be corresponding column of the inverse matrix. Here also we can solve all the systems simultaneously by forming the augmented system.

$$[A/I] = \left(\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right)$$

SOME SOLVED EXAMPLES

Example 3.62. *Find the inverse of* $A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$ *by Gauss-Jordan method.*

Solution. The augmented system $[A/I]$ is

$$[A/I] = \left(\begin{array}{ccc|ccc} 3 & -3 & 4 & 1 & 0 & 0 \\ 2 & -3 & 4 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$R_1 \to \frac{1}{3}R_1 \sim \left(\begin{array}{ccc|ccc} 1 & -1 & \frac{4}{3} & \frac{1}{3} & 0 & 0 \\ 2 & -3 & 4 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$R_2 \to R_2 - 2R_1 \sim \left(\begin{array}{ccc|ccc} 1 & -1 & \frac{4}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -1 & \frac{4}{3} & -\frac{2}{3} & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$R_2 \to (-1)R_2 \sim \left(\begin{array}{ccc|ccc} 1 & -1 & \frac{4}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -\frac{4}{3} & \frac{2}{3} & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$R_1 \to R_1 + R_2 \text{ and } R_3 \to R_3 + R_2 \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -\frac{4}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 1 \end{array}\right)$$

$$R_3 \to (-3)\, R_3 \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -\frac{4}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 0 & 1 & -2 & 3 & -3 \end{array}\right)$$

$$R_2 \to R_2 + \frac{4}{3} R_3 \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 3 & -4 \\ 0 & 0 & 1 & -2 & 3 & -3 \end{array}\right)$$

Inverse of $A = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{pmatrix}$

EXERCISE 3.12

1. Find the inverse of the following matrices by Gauss-Jordan method:

i. $\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ ii. $\begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$ iii. $\begin{pmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{pmatrix}$ iv. $\begin{pmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}$

v. $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 3 & -1 & -4 \end{pmatrix}$

Answers

1. i. $\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$ ii. $\begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & 4 \\ -2 & 3 & -3 \end{pmatrix}$ iii. $\frac{1}{8}\begin{pmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{pmatrix}$ iv. $\begin{pmatrix} -\frac{4}{3} & 2 & \frac{7}{3} \\ \frac{5}{3} & -2 & -\frac{8}{3} \\ \frac{7}{3} & -3 & -\frac{10}{3} \end{pmatrix}$

v. $\begin{pmatrix} \frac{8}{3} & -1 & \frac{2}{3} \\ -\frac{4}{3} & 1 & -\frac{1}{3} \\ \frac{7}{3} & -1 & \frac{1}{3} \end{pmatrix}$

INTERESTING FACTS

- It has a unique application in the domain of environmental sciences, like, in determining average caloric value of specific fishes, growth of forest and its types, etc. For example, if there are 5 different types of trees in a forest, we can create a linear equation to measure their age. Similarly, we can create linear equations to find the intake of carbohydrates of marine life based on their diet, and there are many such examples.
- In real world, we can apply this in ***Traffic Control Management*** to tackle the issue of Traffic jams using ***Gauss Jordan method***, involving the technique of finding inverse of a matrix, by forming ***neutrosophic linear equations*** (which revolves around unrealistic dataset and represents determinate and/or indeterminate information)and applying MATLAB programming.

VIDEO REFERENCES

Gauss Elimination

Gauss-Jordon Method

System of Linear Equations –Gauss Elimination

APPLICATIONS TO REAL LIFE

- Finding the values of unknown quantities; let it be age, cost, or any other thing.
- At airports, where high-end computers are used to calculate and encode information about flights, passengers, etc.
- In circuit analysis, Gauss Jordan process is used on mesh-connected processors.
- It is used in scheduling algorithms.
- Useful in Fingerprint Image Enhancement, which involves the application of Gaussian Elimination method.

SUBJECTIVE SOLVED QUESTIONS
(HOTS)

Example 1. *Determine the number of values of k for which the system of equation*

$$(k + 1)x + 8y = 4k$$

$$kx + (k + 3)y = 3k - 1$$

has infinitely many solution.

Solution. Given system of linear equation is

$$(k+1)x + 8y = 4k$$
$$kx + (k+3)y = 3k - 1$$

It can be written in the form $AX = B$,

where $$A = \begin{bmatrix} k+1 & 8 \\ k & k+3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$$

and $$B = \begin{bmatrix} 4k \\ 3k-1 \end{bmatrix}$$

Now, $$[A : B] = \begin{bmatrix} k+1 & 8 & : & 4k \\ k & k+3 & : & 3k-1 \end{bmatrix}$$

Operating $$R_2 \to R_2 - \frac{k}{k+1} R_1$$

$$[A : B] \sim \begin{bmatrix} k+1 & 8 & : & 4k \\ 0 & (k+3) - \dfrac{8k}{k+1} & : & (3k-1) - \dfrac{4k^2}{k+1} \end{bmatrix}$$

$$\sim \begin{bmatrix} k+1 & 8 & : & 4k \\ 0 & \dfrac{k^2 - 4k + 3}{k+1} & : & \dfrac{-k^2 + 2k - 1}{k+1} \end{bmatrix}$$

It is given that system of equation has infinitely many solution

$\therefore$ $$\rho(A) = \rho(A : B) < n \ (= 2)$$

For this, $$\frac{k^2 - 4k + 3}{k+1} = 0 \quad ...(1)$$

and $$\frac{-k^2 + 2k - 1}{k+1} = 0 \quad ...(2)$$

From (1), $k^2 - 4k + 3 = 0$

$\Rightarrow$ $(k-3)(k-1) = 0$

$\Rightarrow$ $k = 3, 1$

From (2), $-k^2 + 2k - 1 = 0$

$\Rightarrow$ $k^2 - 2k + 1 = 0$

$\Rightarrow$ $(k-1)^2 = 0$

$\Rightarrow$ $k = 1$

$\therefore$ $k = 1$ is the only solution for which system of equation has infinitely many solution.

Example 2. *Prove that the determinant* $\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix}$ *= 0 if a, b, c are in G.P.*

Solution. Let,
$$\Delta = \begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix}$$

Operating $R_3 \to R_3 - (\alpha R_1 + R_2)$
$$= \begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ 0 & 0 & -a\alpha^2 - 2b\alpha - c \end{vmatrix}$$

Expanding along R_3, we get
$$\Delta = 0\begin{vmatrix} b & a\alpha + b \\ c & b\alpha + c \end{vmatrix} - 0\begin{vmatrix} a & a\alpha + b \\ b & b\alpha + c \end{vmatrix} - (a\alpha^2 + 2b\alpha + c)\begin{vmatrix} a & b \\ b & c \end{vmatrix}$$
$$= -(a\alpha^2 + 2b\alpha + c)\,(ac - b^2)$$
$$= (b^2 - ac)\,(a\alpha^2 + 2b\alpha + c)$$

Thus, $\Delta = 0$ if either $b^2 - ac = 0$

or $a\alpha^2 + 2b\alpha + c = 0$

But it is given that

$\therefore$ $b^2 = ac$

or $\dfrac{b}{a} = \dfrac{c}{b}$

i.e., a, b, c are in G.P. Hence, $\Delta = 0$.

Example 3. *If* $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$; $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *and* $A^{-1} = \dfrac{1}{6}(A^2 + cA + dI)$ *where* $c, d \in R$, *then find the value of* (c, d).

Solution. We have $|A| = 1(4 + 2) - 0 + 0 = 6$
$$A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{1}{6}\begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

Now,
$$A^2 = A.A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix}$$

Also,
$$cA = \begin{bmatrix} c & 0 & 0 \\ 0 & c & c \\ 0 & -2c & 4c \end{bmatrix}$$

and $$dI = \begin{bmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix}$$

Given that $$A^{-1} = \frac{1}{6}(A^2 + cA + dI)$$

$$\frac{1}{6}\begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 2 & 1 \end{bmatrix} = \frac{1}{6}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix} + \begin{bmatrix} c & 0 & 0 \\ 0 & c & c \\ 0 & -2c & 4c \end{bmatrix} + \begin{bmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix}\right)$$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+c+d & 0 & 0 \\ 0 & -1+c+d & 5+c \\ 0 & -10-2c & 14+4c+d \end{bmatrix}$$

By equality of matrices, equating corresponding elements, we get

$$6 = 1 + c + d \quad \Rightarrow \quad 5 = c + d$$

$$-1 = 5 + c \quad \Rightarrow \quad -6 = c$$

So, $$5 = -6 + d \quad \Rightarrow \quad d = 11$$

So, $(-6, 11)$ is the required value.

Example 4. *Consider matrix* $A = \begin{bmatrix} k & 2k \\ k^2 - k & k^2 \end{bmatrix}$ *and vector* $X = [X_1\ X_2]^T$.

Find the number of distinct real values of k for which the equation AX = 0 has infinitely many solution.

Solution. The given system has infinitely many solution.

$\therefore$ $$|A| = 0$$

or $$\begin{vmatrix} k & 2k \\ k^2 - k & k^2 \end{vmatrix} = 0$$

i.e., $$k^3 - 2k(k^2 - k) = 0$$

$$k^3 - 2k^3 + 2k^2 = 0$$

$$-k^3 + 2k^2 = 0$$

$$k^2(-k + 2) = 0$$

$$k = 0 \quad \text{or} \quad k = 2$$

Hence, k has two values for which given system of linear equation has infinitely many solution.

Example 5. *The matrix* $A = \begin{bmatrix} a & 0 & 3 & 7 \\ 2 & 5 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & b \end{bmatrix}$ *has det. A = 100 and trace A = 14. Find the value of* $|a - b|$.

Solution. Given that trace $(A) = 14$

$\Rightarrow$ $$a + 5 + 2 + b = 14$$

$$a + 7 + b = 14$$

$$a + b = 7 \qquad ...(1)$$

Also, $\quad$ det. $A = 100$

Expanding about R_4,

$$b\begin{vmatrix} a & 0 & 3 \\ 2 & 5 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 100$$

$\Rightarrow \quad b[a(10 - 0) - 0 + 3(0)] = 100$

$\Rightarrow \quad 10ab = 100$

$\Rightarrow \quad ab = 10$

$$b = \frac{10}{a} \qquad ...(2)$$

Putting the value of b from (2) in (1), we get

$$a + \frac{10}{a} = 7$$

$\Rightarrow \quad a^2 + 10 = 7a$

$\Rightarrow \quad a^2 - 7a + 10 = 0$

$\Rightarrow \quad (a - 5)(a - 2) = 0$

$\Rightarrow \quad a = 5$ or 2

From (2), $\quad b = 2$ or 5

Now, $\quad |a - b| = |5 - 2|$ or $|2 - 5|$

$= 3.$

Example 6. *Consider the matrix* $J_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ *which is obtained by reversing the order of the columns of the identity matrix* I_6. *Let* $P = I_6 + \alpha J_6$ *where* $\alpha \geq 0$, $\alpha \in R$. *Find the value of* α *for which* $\det(P) = 0$.

Solution. Let $\quad P = I_2 + \alpha J_2$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}$$

$\therefore \quad |P| = 1 - \alpha^2$

det. $P = I_4 + \alpha J_4$

$$= \begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & \alpha & 0 \\ 0 & \alpha & 1 & 0 \\ \alpha & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \qquad |P| = \begin{vmatrix} 1 & \alpha & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - 0 + 0 - \alpha \begin{vmatrix} 0 & 1 & \alpha \\ 0 & \alpha & 1 \\ \alpha & 0 & 0 \end{vmatrix}$$

$$= 1[1(1-0) - \alpha(\alpha - 0) + 0] - \alpha[\alpha(1-\alpha^2)]$$

$$= 1 - \alpha^2 - \alpha(\alpha - \alpha^3)$$

$$= 1 - \alpha^2 - \alpha^2 + \alpha^4$$

$$= 1 - 2\alpha^2 + \alpha^4$$

$$= (1-\alpha^2)^2$$

Similarly if $P = I_6 + \alpha J_6$, then

$$|P| = (1-\alpha^2)^3$$

$$\det. P = 0 \quad \Rightarrow \quad (1-\alpha^2)^3 = 0$$

$$\Rightarrow \qquad 1 - \alpha^2 = 0$$

$$\Rightarrow \qquad (1-\alpha)(1+\alpha) = 0$$

$$\Rightarrow \qquad \alpha = -1, 1$$

$\because$ α is non-negative.

$$\therefore \qquad \alpha = 1$$

Example 7. *Let A be $m \times n$ matrix and B be $n \times m$ matrix. It is given that det $(I_m + AB)$ = det $(I_n + BA)$, where I_k is the $k \times k$ identity matrix. Using the above property, calculate determinant of matrix* $\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

Solution. Given

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Let $$AB = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} * [1 \;\; 1 \;\; 1 \;\; 1] = A * B$$

and
$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4 \times 4}$$

$\therefore$ $|I_4| = 1$

It is given that, $|I_m + AB| = |I_n + BA|$

$\therefore$
$$BA = [1 \;\; 1 \;\; 1 \;\; 1] * \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = [4]$$

$$\det(I_n + BA) = |[4] + [1]|$$
$$= |5| = 5$$

Example 8. *Find the maximum value of the determinant among all 2 × 2 real symmetric matrices with trace 14.*

Solution. General 2 × 2 real symmetric matrix is

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

where $|A| = ac - b^2$ and

$$\text{trace } A = a + c = 14 \quad \ldots(1)$$

For maximum value of $|A|$, b^2 must be minimum.

Since b^2 is always non-negative number so the minimum value is

$$b^2 = 0$$

i.e., $b = 0$ for maximum determinant.

$$|A| = ac = a(14 - a) \quad \text{[from (1)]}$$
$$= 14a - a^2$$

For maximum value of $|A|$, we may write

$$\frac{d|A|}{da} = 0$$

$\Rightarrow$ $14 - 2a = 0$

$$a = 7$$

From (1), $c = 14 - a$

$\Rightarrow$ $c = 7$

Maximum determinant is

$$|A| = ac = 7 \times 7 = 49.$$

Example 9. *There are two families. Family P consists of 3 men, 3 women and 12 children. Family Q consists of 2 men, 2 women and 4 children. The recommended daily allowance for calories is man : 2400, woman : 2000, child : 1400 and for proteins is man : 60 g, woman : 40 g and child : 35 g. Using matrix multiplication, calculate the total requirement of calories and proteins of each of the two families.*

Solution. Let A be the matrix representing the number of men, women and children in families P and Q. Then A can be written as

$$A = \begin{matrix} \\ P \\ Q \end{matrix}\begin{bmatrix} \text{Men} & \text{Women} & \text{Children} \\ 3 & 3 & 12 \\ 2 & 2 & 4 \end{bmatrix}$$

Let B be the matrix representing the recommended daily allowance for calories and proteins. Then B can be written as

$$B = \begin{matrix} \text{Man} \\ \text{Woman} \\ \text{Child} \end{matrix}\overset{\begin{matrix}\text{Calories} & \text{Proteins}\end{matrix}}{\begin{bmatrix} 2400 & 60 \\ 2000 & 40 \\ 1400 & 35 \end{bmatrix}}$$

The total requirement of calories and proteins for families P and Q are given by the product of two matrices *i.e.*,

$$AB = \begin{bmatrix} 3 & 3 & 12 \\ 2 & 2 & 4 \end{bmatrix}\begin{bmatrix} 2400 & 60 \\ 2000 & 40 \\ 1400 & 35 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \times 2400 + 3 \times 2000 + 12 \times 1400 & 3 \times 60 + 3 \times 40 + 12 \times 35 \\ 2 \times 2400 + 2 \times 2000 + 4 \times 1400 & 2 \times 60 + 2 \times 40 + 4 \times 35 \end{bmatrix}$$

$$= \begin{bmatrix} 7200 + 6000 + 16800 & 180 + 120 + 420 \\ 4800 + 4000 + 5600 & 120 + 80 + 140 \end{bmatrix}$$

$$= \begin{matrix} \\ P \\ Q \end{matrix}\begin{bmatrix} \text{Calories} & \text{Protiens} \\ 30000 & 720 \\ 14400 & 340 \end{bmatrix}$$

Hence, the total requirement of calories and proteins for family P are 30000 calories and 720 g proteins respectively and for family Q are 14400 calories and 340 g proteins respectively.

SUMMARY

1. Matrix is an array representation of $(m \times n)$ elements and write as

$$A = [a_{ij}]_{m \times n},\ i = 1, 2, ..., m$$
$$j = 1, 2, ..., n$$

2. **Idempotent Matrix:** A square matrix 'A' is said to be idempotent if $A^2 = A$.

3. **Involutory Matrix:** A square matrix 'A' is said to be involutary if $A^2 = I$. Unit matrix is always an involutary matrix.
4. **Rank:** Number of non-zero rows in an echelon form of the matrix is called the rank of matrix.
5. For a non-homogeneous system
   a. If $\rho(A) = \rho(A : B)$ = number of unknowns, then the system is consistent with unique solution.
   b. If $\rho(A) = \rho(A : B)$ < number of unknowns, then system is consistent with infinite many solutions.
   c. If $\rho(A) \neq \rho(A : B)$, then system is inconsistent.
6. For a homogeneous system
   a. If $\rho(A)$ = number of unknowns, then system is consistent with unique solution (trivial solution).
   b. If $\rho(A)$ < number of unknowns, then system is consistent with infinite solutions (non-trivial solutions).
7. If determinant = 0, then Cramer's is rule can not apply.
8. If the determinant of a matrix is non-zero, then its inverse exists and it is always unique.

OBJECTIVE QUESTIONS

1. If $2\begin{bmatrix} x & 9 \\ y & 6 \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$, then the values of x and y are

   a. $x = 6, y = 3$ b. $y = 6, x = 3$ c. $x = 9/2, y = 6$ d. $x = -1, y = -2$

2. If $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, then the value of the matrix X is,

   a. $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}$ c. $\begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$ d. $\begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}$

3. If $A = [2, 1, -3]$ and $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then the product BA is

   a. 15 b. $\begin{bmatrix} 2 & 1 & -3 \\ 4 & 2 & -6 \\ 6 & 3 & -9 \end{bmatrix}$ c. $\begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ -3 & -6 & -9 \end{bmatrix}$ d. 26

4. The values of y for which the following matrices are equal

$$\begin{bmatrix} 3x+7 & 5 \\ x+y & 3 \end{bmatrix} = \begin{bmatrix} 7 & 5 \\ 4 & 3 \end{bmatrix} \text{ is}$$

   a. 4 b. 3 c. 0 d. –1

5. If $A = \begin{bmatrix} 6 & -1 \\ 2 & 5 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 \\ 0 & 4 \end{bmatrix}$, then

   a. AB exists b. BA exists c. $(A + B)$ exists d. $A - B$ exists

6. Let A be a square matrix of order 3, then $|kA|$ is equal to

a. $3k|A|$ b. $k|A|$ c. $k^2|A|$ d. $k^3|A|$

7. If $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$, then value of x is

a. 3 b. 4 c. 2 d. –1

8. If A is an invertible matrix of order 2, then det (A^{-1}) is equal to

a. 0 b. det A c. 1 d. $\dfrac{1}{\det A}$

9. The value of the determinant $\begin{vmatrix} 0 & 9 & 12 \\ 1 & -3 & -4 \\ 1 & 9 & 12 \end{vmatrix}$ is

a. 1 b. –1 c. 0 d. 2

10. If $\begin{vmatrix} x+2 & 3 \\ x+5 & 4 \end{vmatrix} = 3$, then the value of x is,

a. 7 b. 8 c. 12 d. 10

11. If a matrix A is both symmetric and skew-symmetric, then

a. A is a diagonal matrix b. A is a zero matrix

c. A is a scalar matrix d. A is a square matrix

12. If A = $\begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$ and $a^2 + b^2 + c^2 + d^2 = 1$, then A^{-1} is

a. $\begin{bmatrix} a+ib & -c+id \\ -a+id & a-ib \end{bmatrix}$ b. $\begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$ c. $\begin{bmatrix} a-ib & c-id \\ -c-id & a+ib \end{bmatrix}$ d. $\begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$

13. If $A(\alpha, \beta) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & e^{\beta} \end{bmatrix}$, then $A(\alpha, \beta)^{-1}$ is

a. $A(-\alpha, \beta)$ b. $A(-\alpha, -\beta)$ c. $A(\alpha, -\beta)$ d. $A(\alpha, \beta)$

14. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, such that $A^{-1} = kA$, then the value of k is

a. $\dfrac{1}{19}$ b. $-\dfrac{1}{19}$ c. $\dfrac{1}{17}$ d. $-\dfrac{1}{17}$

15. The rank of matrix $\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{bmatrix}$, a, b, c being real, is 3, then

a. $a = b = c$

b. a, b, c are all different but $a + b + c = 0$

c. two of the numbers a, b, c are equal but are different from the third

d. a, b, c are all different and $a + b + c \neq 0$

16. Find the value of p for which, the rank of the given matrix is 1.

$$\begin{bmatrix} 3 & p & p \\ p & 3 & p \\ p & p & 3 \end{bmatrix}$$

a. 4 b. 2 c. 3 d. 1

17. Solve the following equations using Gauss elimination method

$$x + 2y + 3z = 4,\ 2x + 3y + 4z = 5,\ 3x + 4y + 5z = 6$$

a. $x = 0.5, y = 0$ and $z = 1$ b. $x = 0.5, y = 0$ and $z = -1$

c. $x = -0.5, y = 0$ and $z = 1.5$ d. $x = 0.5, y = 0$ and $z = -1.5$

18. In Gauss Jordon method which of the following transformations are allowed?

a. Diagonal transformation b. Column transformation

c. Row transformation d. Square transformation

19. Apply Cramer's rule to solve the following equations

$$3x + y + 2z = 3,\ 2x - 3y - z = -3,\ x + 2y + z = 4$$

a. $x = 1, y = 2, z = -1$ b. $x = 2, y = 1, z = -1$

c. $x = 2, y = -1, z = 1$ d. $x = 1, y = -1, z = 2$

20. Cramer's rule fails for

a. determinant > 0 b. determinant < 0

c. determinant = 0 d. determinant = non-real

Answers

1. b	**2.** c	**3.** b	**4.** b
5. a	**6.** d	**7.** c	**8.** d
9. c	**10.** d	**11.** b	**12.** c
13. b	**14.** a	**15.** d	**16.** c
17. c	**18.** c	**19.** a	**20.** c

SUBJECTIVE UNSOLVED QUESTIONS
(HOTS)

1. Show that if A has a zero row, then AB also has a zero row.
2. Show that if B has a zero column, then AB also has a zero column.
3. Let A be an $m \times n$ matrix with rank m and B be an $n \times p$ matrix with rank n. Determine the rank of AB. Justify your answer.
4. Prove that if B is a 3×1 matrix and C is a 1×3 matrix, then the 3×3 matrix BC has rank at most 1. Conversely, show that if A is any 3×3 matrix having rank 1, then there exist a, 3×1 matrix B and 1×3 matrix C, such that $A = BC$.
5. Find 2×2 invertible matrices A and B such that $A + B$ is not equal to zero and $A + B$ is not invertible.
6. Let $A \in M_{n\times n}(F)$. Under what conditions, $\det(-A) = \det(A)$.

7. A certain economy consists of two sectors: goods and services. Suppose that 60% of all goods and 30% of all services are used in the production of goods. What proportion of the total economic output is used in the production of goods?

8. Give a counter example to the following statement: If the co-efficient matrix of a system of *m* linear equations in n unknowns has rank m, then the system has a solution.

PROJECT/PRACTICAL/ACTIVITIES

PROJECT

Prepare a model showing various types of matrices and their applications in graph theory

PRACTICAL

1. Write a MATLAB function that takes a matrix, a row number and a column number. Beginning with the row number passed to the function, scroll down the column passed to the function and return the row number that contains the largest absolute value in the column.

2. Using MATLAB, find the determinant of the 3 × 3 matrix.

ACTIVITY

1. A shopkeeper sells packets P_1 of 1 kg. of wheat, 1 kg of rice and 1 kg of Bajra and P_2 containing of 1 kg. of wheat, 0 kg of rice and 1 kg of Bajra and P_3 comprising of 0 kg. of wheat, 1 kg of rice and 1 kg of Bajra.
Check, Is it possible to buy only one kg. of Bajra?
If Yes, How?

2. What is a vector? Is it a Type of Matrix? Think and support your answer by giving an example.

3. Form a group of students from various cities, make a graph and form adjacency matrix for the same with vertices as cities and edges as transportation cost of a good.

KNOW MORE

1. The least value of the product *xyz* for which the determinant $\begin{vmatrix} x & 1 & 1 \\ 1 & y & 1 \\ 1 & 1 & z \end{vmatrix}$ is non-negative is

a. -8 b. -1 c. $-2\sqrt{2}$ d. $-16\sqrt{2}$

2. If $A = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$ is an orthogonal matrix, then the number of possible triplets (α, β, γ) is

a. 8 b. 6 c. 4 d. 2

3. If $\alpha, \beta \neq 0$ and $f(n) = \alpha^n + \beta^n$ and $\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix} = k(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$, then k is equal to

a. $\alpha\beta$ b. $1/\alpha\beta$ c. 1 d. –1

4. Find the rank of $A = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$.

5. Find 'a' so that the rank of the matrix $A = \begin{bmatrix} a & 0 & 1 \\ 1 & 2 & a \\ 1 & 2 & 3 \end{bmatrix}$ is less than 3.

Answers

1. a **2.** a **3.** c

4. $\rho(A) = \begin{cases} 3 & \text{if } x \neq y, y \neq z, z \neq x \\ 2 & \text{if either } x = y \quad \text{or } x = z \text{ and } y \neq z \\ 1 & \text{if } x = y = z \end{cases}$ **5.** 0, 3

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4 Vector Spaces I

UNIT SPECIFICS

This unit elaborately explains the following topics - vector space, linear dependence and independence of vectors, linear combination, linear span basis, dimension, linear transformations (maps), range and kernel of a linear map, rank and nullity, inverse of a linear transformation, rank-nullity theorem, composition of linear maps, matrix associated with a linear map. All the concepts have been connected with ample examples to make theory and application part more clear to the students.

RATIONALE

Linear algebra is applied everywhere, but it can be hard to see sometimes. Today, vector spaces are applied throughout mathematics, science and engineering. They are the appropriate linear-algebraic notion to deal with systems of linear equations. A linear transformation is not an application in itself; rather, it is a model. Linear transformation is one of the core content of linear algebra. Concept of linear transformation is the transformation of coordinates in analytical geometry. Its theory and methods lies in analytical geometry, differential equations and many other fields and it has widespread application also. Matrices could be used to encode messages, and the decoder is the inverse of the matrix. A matrix is a linear map between vector spaces and we can use certain matrices to study rotations in a plane.

PRE-REQUISITES

1. Good understanding of rank, inverse and determinant of matrix.
2. Command on formation and analysis of solution of system of linear equation.
3. Student should know how to apply elementary operations on matrix.
4. Clear understanding about the behaviour of function such as surjective, injective etc.

UNIT OUTCOMES

After completion of this unit, students will be able to:

U4-01: Understand the concepts of vector spaces, subspaces, bases, dimension and their properties.

U4-02: Learn about properties of linear transformations; interpret the correlation of matrices with linear transformations from R^n to R^m and vice-versa.

U4-03: Examine and determine, how the matrices change, when their bases are changed; learn about Rank-Nullity Theorem.

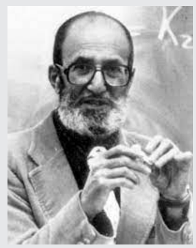
U4-04: Evaluate the concept of range spaces, null space, kernel space and special types of linear transformations; relate basis/linear maps with matrices and vice-versa.

MAPPING OF UNIT OUTCOMES WITH COURSE OUTCOMES

Unit 4 Outcomes	EXPECTED MAPPING WITH COURSE OUTCOMES (1- Weak Correlation; 2- Medium Correlation; 3- Strong Correlation)				
	CO-1	CO-2	CO-3	CO-4	CO-5
U4-01	–	–	–	3	1
U4-02	–	–	2	3	2
U4-03	–	–	1	3	2
U4-04	–	–	2	3	2

HISTORY

The idea of a vector space developed from the notion of ordinary two- and three-dimensional spaces as collections of vectors $\{u, v, w, \ldots\}$ with an associated field of real numbers $\{a, b, c, \ldots\}$. Vector spaces as abstract algebraic entities were first defined by the Italian mathematician Giuseppe Peano in 1888. Peano called his vector spaces "linear systems" because he correctly saw that one can obtain any vector in the space from a linear combination of finitely many vectors and scalars—$av + bw + \ldots + cz$.



"The only way to learn mathematics is to do mathematics."

—Paul Halmos

4.1 VECTOR SPACE

Let F be a field and a non-empty set V together with two binary operations called vector addition '+' and scalar multiplication ' . ' is called vector space over field F if this structure satisfies the following conditions.

i. V is closed under addition *i.e.* $u + v \in V$ for all $u, v \in V$.
ii. Associativity, means $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$.
iii. **Existence of additive identity:** There exists an element $0 \in V$ such that $u + 0 = 0 + u = u$ for all $u \in V$.
iv. **Existence of additive inverse:** For each $u \in V$, there exists a unique element $-u \in V$ such that $u + (-u) = 0 = (-u) + u$.
v. **Commutativity:** $u + v = v + u$ for all $u, v \in V$.
vi. V is closed under scalar multiplication; *i.e.*, $au \in V$ for all $a \in F, u \in V$.
vii. $a(u + v) = au + av$ for all $a \in F, u, v \in V$.
viii. $(a + b)u = au + bu$ for all $a, b \in F, u \in V$.

ix. $(ab)u = a(bu)$ for all $a, b \in F, u \in V$.

x. $1u = u$ for all $u \in V$ and '1' is multiplicative identity of F.

Hence, $V(F)$ is a vector space.

> **Remark:** Elements of V are called vectors and elements of F are called scalars.

For example. *Check Z(Q) is a vector space or not?*

Solution. Let $1 \in Z$ and $\frac{1}{2} \in Q$.

By Property (vi), $au \in V$ for all $a \in F$ and $u \in V$.

$$\therefore \qquad 1 \cdot \frac{1}{2} = \frac{1}{2} \notin Z$$

Thus, $Z(Q)$ is not a vector space.

4.1.1 Vectors in $\mathbb{R}^n$

A vector in n (real) dimensional is defined to be an ordered n-tuple (a_1, a_2, a_n), where each of the a_i is real number $(a_i \in \mathbb{R})$. The set of all n is dimensional vectors is denoted by $\mathbb{R}^n$. An ordered n-tuple of real numbers is called real n-vectors. Mathematically, it is written as

Mathematically, $\qquad \mathbb{R}^n = \{a_1, a_2, ..., a_n \mid a_i \in \mathbb{R}, 1 \le i \le n\}$.

4.1.2 Vectors in Matrices

$$M_{m\times n}(F) = \text{set of all } m \times n \text{ matrices whose entries are from } F$$

$$= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, a_{ij} \in F$$

4.1.3 Vectors in Polynomial of Degree atmost *n*

$$P_n(F) = \text{set of all the polynomials of degree atmost } n \text{ with coefficient from } F \text{ and zero polynomial}$$

$$= \{a_0 + a_1 x + ... + a_n x^n, a_i \in F, n \in N \cup \{0\}\}$$

SOME SOLVED EXAMPLES

Example 4.1. *Prove that the set of all diagonal matrices over* **R** *is a vector space with respect to matrix addition and scalar multiplication.*

Solution. Let $\qquad V = \text{set of all diagonal matrices of order } n \times n$

$$= [a_{ii}]_{n\times n}, a_{ii} \in \mathbf{R}$$

As we know that matrix addition of same order $n \times n$ is defined and matrix multiplication is

$$\alpha A = [\alpha\, a_{ii}]_{n\times n}$$

i. Let $[a_{ii}]_{n\times n}, [b_{ii}]_{n\times n} \in V$, then

$$[aii]_{n\times n} + [b_{ii}]_{n\times n} = [a_{ii} + b_{ii}]_{n\times n}$$

is also a diagonal matrix.

ii. Let $[a_{ii}]_{n\times n}, [b_{ii}]_{n\times n}, [c_{ii}]_{n\times n} \in V$

$$\begin{aligned}
\therefore \quad [a_{ii}]_{n\times n} + \{[b_{ii}]_{n\times n} + [c_{ii}]_{n\times n}\} &= [a_{ii}]_{n\times n} + [b_{ii} + c_{ii}]_{n\times n} \\
&= [a_{ii}]_{n\times n} + [b_{ii} + c_{ii}]_{n\times n} \\
&= [a_{ii} + (b_{ii} + c_{ii})]_{n\times n} \\
&= [(a_{ii} + b_{ii}) + c_{ii}]_{n\times n} \qquad [\because \text{ addition is associative in } R'] \\
&= [a_{ii} + b_{ii}]_{n\times n} + [c_{ii}]_{n\times n} \\
&= [a_{ii}]_{n\times n} + [b_{ii}]_{n\times n} + [c_{ii}]_{n\times n}
\end{aligned}$$

iii. Let $A = [aii]_{n\times n} \in V.$

We know that zero matrix of order $n \times n$ belong to set V and is denoted by $\mathbf{0}$.

$$\begin{aligned}
\mathbf{0} &= [0]_{n\times n} \\
A + \mathbf{0} &= [a_{ii}]_{n\times n} + [0]_{n\times n} \\
&= [a_{ii} + 0]_{n\times n} \\
&= [a_{ii}]_{n\times n} = A
\end{aligned}$$

Similarly, $\mathbf{0} + A = A$

iv. Let $A = [a_{ii}]_{n\times n} \in V$, then there exists

$$-A = [-a_{ii}]_{n\times n} \in V$$

Now,

$$\begin{aligned}
A + (-A) &= [a_{ii}]_{n\times n} + [-a_{ii}]_{n\times n} \\
&= [a_{ii} - a_{ii}]_{n\times n} \\
&= [0]_{n\times n} \\
&= \mathbf{0}
\end{aligned}$$

Similarly, $(-A) + A = \mathbf{0}$

v. Let $A = [a_{ii}]_{n\times n}, B = [b_{ii}]_{n\times n} \in V$

$$\begin{aligned}
\therefore \quad A + B &= [a_{ii}]_{n\times n} + [b_{ii}]_{n\times n} \\
&= [a_{ii} + b_{ii}]_{n\times n} \\
&= [b_{ii} + a_{ii}]_{n\times n} \qquad [\because \text{ addition is commutative in } R] \\
&= [b_{ii}]_{n\times n} + [a_{ii}]_{n\times n} \\
&= B + A
\end{aligned}$$

vi. Let $a \in \mathbf{R}, A = [a_{ii}]_{n\times n} \in V$

Now,

$$\begin{aligned}
aA &= a[a_{ii}]_{n\times n} \\
&= [aa_{ii}]_{n\times n} \\
&= \text{diagonal matrix of order } n \times n.
\end{aligned}$$

vii. Let $A = [a_{ii}]_{n\times n}, B = [b_{ii}]_{n\times n} \in V$ and $a \in \mathbf{R}$

$$\begin{aligned}
\therefore \quad a(A + B) &= a([a_{ii}]_{n\times n} + [b_{ii}]_{n\times n}) \\
&= a[a_{ii} + b_{ii}]_{n\times n} = [a(a_{ii} + b_{ii})]_{n\times n} \\
&= [a \,.\, a_{ii} + ab_{ii}]_{n\times n} \qquad [\because \text{ Multiplication is distributive in } R] \\
&= [a \,.\, a_{ii}]_{n\times n} + [a \,.\, b_{ii}]_{n\times n} \\
&= a[a_{ii}]_{n\times n} + a[b_{ii}]_{n\times n} \\
&= aA + aB
\end{aligned}$$

viii. Let $a, b \in \mathbf{R}$ and $A = [a_{ii}]_{n\times n} \in V$

$$\therefore \quad (a+b)A = (a+b)[a_{ii}]_{n\times n}$$
$$= [(a+b)\,.\,a_{ii}]_{n\times n}$$
$$= [a\,.\,a_{ii} + b\,.\,a_{ii}]_{n\times n}$$
$$= [a\,.\,a_{ii}]_{n\times n} + [b\,.\,a_{ii}]_{n\times n}$$
$$= a[a_{ii}]_{n\times n} + [b\,.\,a_{ii}]_{n\times n}$$
$$= a[a_{ii}]_{n\times n} + b\,.\,[a_{ii}]_{n\times n}$$
$$= aA + bA.$$

ix. Let $a, b \in \mathbf{R}$ and $A = [a_{ii}]_{n\times n} \in V$

$$\therefore \quad (ab)A = (ab)[a_{ii}]_{n\times n}$$
$$= [(ab)\,a_{ii}]_{n\times n}$$
$$= [a(ba_{ii})]_{n\times n}$$
$$= a[b\,.\,a_{ii}]_{n\times n}$$
$$= a\,([b[a_{ii}]_{n\times n})$$
$$= a(bA)$$

x. Let $A = [a_{ii}]_{n\times n} \in V$, then

$$1\,.\,A = 1\,.\,[a_{ii}]_{n\times n} = [1\,.\,a_{ii}]_{n\times n}$$
$$= [a_{ii}]_{n\times n} = A$$

Thus V satisfies all the properties of vector space and hence $V(\mathbf{R})$ is a vector space.

Example 4.2. *Show that the set M of all m × n matrices with their elements as real no. is a vector space over the field R of real numbers w.r.t. addition of matrix addition of vectors and multiplication of a matrix by a scalar as scalar multiplication.*

Solution. $M = \{[a_{ij}]_{m\times n};\, a_{ij} \in R\}$

Matrix addition of same order $m \times n$ is defined and also matrix multiplication is also defined.

i. Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n} \in M$, then

$$A + B = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}$$
$$= [a_{ij} + b_{ij}]_{m\times n} \in M$$

ii. Let $A = [a_{ij}]_{m\times n} \in M$, $B = [b_{ij}]_{m\times n} \in M$, $C = [c_{ij}]_{m\times n} \in M$

$$\therefore \quad [a_{ij}]_{m\times n} + \{[b_{ij}]_{m\times n} + [c_{ij}]_{m\times n}\}$$
$$= [a_{ij}]_{m\times n} + [b_{ij} + c_{ij}]_{m\times n}$$
$$= [a_{ij} + b_{ij} + c_{ij}]_{m\times n}$$
$$= [(a_{ij} + b_{ij}) + c_{ij}]_{m\times n} \qquad [\because \text{ addition is associative in } R]$$
$$= [a_{ij} + b_{ij}]_{m\times n} + [c_{ij}]_{m\times n}$$
$$= [[a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}\} + [c_{ij}]_{m\times n}$$

iii. Let $A = [a_{ij}]_{m\times n} \in M$

We know that zero matrix denoted by **0** of order $m \times n$ belongs to set M.

i.e., $\quad 0 = [0]_{m\times n}$

Now, $\quad A + 0 = [a_{ij}]_{m\times n} + [0]_{m\times n}$
$$= [a_{ij} + 0]_{m\times n}$$

$$= [a_{ij}]_{m\times n} = A$$

Similarly, $0 + A = A$

iv. Let $A = [a_{ij}]_{m\times n} \in M$, then there exists

$$-A = [-a_{ij}]_{m\times n} \in M$$

Now, $A + (-A) = [a_{ij}]_{m\times n} + [-a_{ij}]_{m\times n}$

$$= [a_{ij} - a_{ij}]_{m\times n}$$
$$= [0]_{m\times n}$$
$$= \mathbf{0}$$

Similarly, $(-A) + A = 0$

v. Let $A = [a_{ij}]_{m\times n} \in M, B = [b_{ij}]_{m\times n} \in M$

$$A + B = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}$$
$$= [a_{ij} + b_{ij}]_{m\times n}$$
$$= [b_{ij} + a_{ij}]_{m\times n} \qquad [\because \text{ addition is commutative in } R]$$
$$= [b_{ij}]_{m\times n} + [a_{ij}]_{m\times n}$$
$$= B + A$$

vi. Let $a \in \mathbf{R}$ and $A = [a_{ij}]_{m\times n} \in M$

Now, $aA = a[a_{ij}]_{m\times n}$

$$= [aa_{ij}]_{m\times n} \in M.$$

vii. Let $a \in \mathbf{R}$ and $A = [a_{ij}]_{m\times n} \in M, B = [b_{ij}]_{m\times n} \in M$

$\therefore$ $a(A + B) = a\{[a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}\}$

$$= a[a_{ij} + b_{ij}]_{m\times n} \qquad [\because \text{ Multiplication is distributive in } R]$$
$$= [a(a_{ij} + b_{ij}]_{m\times n}$$
$$= [a \,.\, a_{ij} + a \,.\, b_{ij}]_{m\times n}$$
$$= [a \,.\, a_{ij}]_{m\times n} + [a \,.\, b_{ij}]_{m\times n}$$
$$= [a \,.\, a_{ij}]_{m\times n} + [a \,.\, b_{ij}]_{m\times n}$$
$$= a[a_{ij}]_{m\times n} + a[b_{ij}]_{m\times n}$$
$$= aA + aB$$

viii. Let $a, b \in \mathbf{R}$ and $A = [a_{ij}]_{m\times n} \in M$

$\therefore$ $(a + b)A = (a + b)\ [a_{ij}]_{m\times n}$

$$= [(a + b) \,.\, a_{ij}]_{m\times n}$$
$$= [a \,.\, a_{ij} + b \,.\, a_{ij}]_{m\times n}$$
$$= [a \,.\, a_{ij}]_{m\times n} + [b \,.\, a_{ij}]_{m\times n}$$
$$= a[a_{ij}]_{m\times n} + b[a_{ij}]_{m\times n}$$
$$= aA + bA.$$

ix. Let $a, b \in \mathbf{R}$ and $A = [a_{ij}]_{m\times n} \in M$

$\therefore$ $(ab)A = (ab)\ [a_{ij}]_{m\times n}$

$$= [(ab)a_{ij}]_{m\times n}$$
$$= [a(b \,.\, a_{ij})]_{m\times n}$$
$$= a[b \,.\, a_{ij}]_{m\times n}$$

$= a\{b[a_{ij}]_{m\times n}\}$
$= a(bA)$

x. Let $A = [a_{ij}]_{m\times n} \in M$, then

$$1 . A = 1 . [a_{ij}]_{m\times n} = [1 . a_{ij}]_{m\times n}$$
$$= [a_{ij}]_{m\times n} = A$$

Thus, M satisfies all the properties of vector space and hence M is a vector space over **R**.

Example 4.3. *Is the set of vector* $(x_1, x_2, x_3, x_4) \in \mathbf{R}^4$ *such that* $x_2 > 0$ *is a vector space for usual addition and scalar multiplication?*

Solution. Let $V = \{(x_1, x_2, x_3, x_4); x_2 > 0 \text{ and } \forall\, x_i \in \mathbf{R}; 1 \le i \le 4\}$

Let $-1 \in R$ and $(x_1, x_2, x_3, x_4) \in V$ where $x_2 > 0$

Now, by property (vi), $au \in V\ \forall\ a \in F$ and $u \in V$

$\Rightarrow \quad -1(x_1, x_2, x_3, x_4) = (-x_1, -x_2, -x_3, -x_4) \notin V$

because $x_2 < 0$

$\therefore \quad V(R)$ is not vector space.

EXERCISE 4.1

1. Prove that

i. **C** is a vector space over **C**

ii. **R** is not a vector space over **C**

iii. **C** is a vector space over Q

iv. Z is not a vector space over Q

v. **C** is a vector space over **R**

2. Let $V = \{(a, b)\ ;\ a, b \in \mathbf{R}\}$ then show that V is not a vector space over real under addition and scalar multiplication defined as in each one of following cases:

i. $(a, b) + (c, d) = (a + c, b + d)$ and $k(a, b) = (0, kb)$

ii. $(a, b) + (c, d) = (0, b + d)$ and $k(a, b) = (ka, kb)$

iii. $(a, b) + (c, d) = (ac, bd)$ and $k(a, b) = (ka, kb)$

iv. $(a, b) + (c, d) = (0, 0)$ and $k(a, b) = (ka, kb)$

3. Which of the following subsets of $\mathbf{R}^4$ are vector spaces for usual addition and scalar multiplication? The set of vectors $(x_1, x_2, x_3, x_4) \in \mathbf{R}^4$ such that

a. $x_1 = 0$ b. $x_1 < 0$ c. $x_4 = 0$ d. $x_3{}^2 \ge 0$

e. $2x_1 + 3x_2 = 0$

4. Prove that the set of all matrices of the form $\begin{bmatrix} x & y \\ -y & x \end{bmatrix}$ where $x, y \in \mathbf{C}$ is a vector space over **C** with respect to matrix addition and scalar multiplication.

Answers

1. a. Yes b. No c. Yes d. Yes

e. Yes

4.2 LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

Linearly dependent vectors: Let $V(F)$ be a vector space. The vectors $\{v_1, v_2, ..., v_n\} \in V$ are said to be linearly dependent (L.D.) if there exist scalars $a_1, a_2, ..., a_n \in F$ (not all zero) such that

$$a_1v_1 + a_2v_2 + ... + a_nv_n = \mathbf{0}$$

Linearly independent vectors: Let $V(F)$ be a vector space. The vectors $\{v_1, v_2, ..., v_n\} \in V$ are said to linearly independent (L.I.) if

$$a_1v_1 + a_2v_2 + ... + a_nv_n = \mathbf{0} \text{ for all } a_i \in F, 1 \leq i \leq n$$

$$\Rightarrow \quad a_1 = a_2 = ... = a_n = 0$$

Results:

i. The empty set is defined to be L.I.
ii. A set containing only zero vector/null vector *i.e.* $\{0\}$ is L.D.
iii. A set containing only one non-zero vector is L.I.
iv. Two vectors are L.D. iff one of them is a scalar multiple of other.

SOME SOLVED EXAMPLES

Example 4.4. *Examine the linear dependence/independence of the following set of vectors:*

i. $\{(1, 2, 3), (1, 0, 0), (0, 2, 3)\}$ in $\mathbf{R}^3$ ii. $\{(1, 1, 1), (1, 2, 3), (0, 1, 3)\}$ in $\mathbf{R}^3$.

Solution. Let $u = (1, 2, 3)$, $v = (1, 0, 0)$, $w = (0, 2, 3)$

Let $au + bv + cw = 0$ for some scalars a, b, c

$\therefore \quad a(1, 2, 3) + b(1, 0, 0) + c(0, 2, 3) = 0$

$\Rightarrow \quad (a + b, 2a + 2c, 3a + 3c) = (0, 0, 0)$

Equating the corresponding elements, we get

$$a + b = 0 \quad ...(1)$$

$$2a + 2c = 0 \quad \Rightarrow \quad a + c = 0 \quad ...(2)$$

$$3a + 3c = 0 \quad \Rightarrow \quad a + c = 0$$

From (1), $a = -b$

From (2), $a = -c$

$\therefore \quad a = -b, a = -c$ is a solution for every value of a.

If $a = 1, b = -1, c = -1$, is a solution. Then, the given set of vectors are L.D.

Alter method: Construct a matrix 'A' whose columns are given vectors.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 3 & 0 & 3 \end{bmatrix}$$

Operating $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{bmatrix}$$

Operating $R_3 \to R_3 - \frac{3}{2} R_2$

$$A \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

This is row-echelon form of matrix A and $\rho(A)$ = rank of $A = 2 <$ No. of column of A.
Hence, given vectors are L.D.

ii. Construct a matrix A by writing vectors in column i.e.,

$$A \sim \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix}$$

Operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

Operating $R_3 \to R_3 - 2R_2$

$$A \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

This is row-echelon form of matrix A and $\rho(A) = 3$ = no. of column of A.

$\therefore$ given vectors are L.I.

Remark: To check whether, vectors are L.D./L.I. by matrix method.

i. Form a matrix A whose columns are given vectors.
ii. Reduce matrix A into row-echelon form by using elementary row operation.
iii. $\rho(A) \begin{cases} = n = \text{no. of column} & \Rightarrow \text{L.I.} \\ < n = (\text{no. of column}) & \Rightarrow \text{L.D.} \end{cases}$

Example 4.5. *Find α if the vectors {(1, –1, 3), (1, 2, –3), (α, 0, 1)} are L.D.*

Solution. Let $u = (1, -1, 3)$, $v = (1, 2, -3)$ and $w = (\alpha, 0, 1)$.

Since vectors are L.D., then

$$au + bv + cw = 0$$

where a, b, c are scalars and not all are zero.

$$a(1, -1, 3) + b(1, 2, -3) + c(\alpha, 0, 1) = 0$$

$$(a + b + \alpha c, -a + 2b, 3a - 3b + c) = (0, 0, 0)$$

Equating the corresponding elements, we get

$$a + b + \alpha c = 0 \quad \text{...(1)}$$

$$-a + 2b = 0 \quad \text{...(2)}$$

$$3a - 3b + c = 0 \quad \text{...(3)}$$

Multiply (2) by 2 and adding with (3)

$$\begin{array}{r} 3a - 3b + c = 0 \\ -2a + 4b \quad\;\; = 0 \\ \hline a + \;\; b + c = 0 \end{array} \qquad \text{...(4)}$$

Subtracting (4) from (1), we get

$$\alpha c - c = 0$$

or $$c(\alpha - 1) = 0$$

But $c \neq 0$, so $\alpha - 1 = 0 \quad \Rightarrow \quad \alpha = 1$

Example 4.6. *Show that the set* $\{x^3 - x + 1, x^3 + 2x + 1, x + 1\}$ *is L.I. set of vectors in the vector space of all polynomial over the field of real numbers.*

Solution. Let $a, b, c \in \mathbf{R}$

$\therefore \quad a(x^3 - x + 1) + b(x^3 + 2x + 1) + c(x + 1) = 0$

$(a + b)x^3 + (-a + 2b + c)x + (a + b + c) = 0$

Equating coefficient of like power of x on both side, we get

$$a + b = 0 \qquad \text{...(1)}$$

$$-a + 2b + c = 0 \qquad \text{...(2)}$$

$$a + b + c = 0 \qquad \text{...(3)}$$

Subtracting (1) from (3), we get

$$c = 0$$

From (2), $\quad -a + 2b = 0 \quad \Rightarrow \quad a = 2b$

From (3), $\quad a + b = 0 \quad \Rightarrow \quad a = -b$

Thus, $\quad -b = 2b \quad \Rightarrow \quad 3b = 0$

$\Rightarrow \quad b = 0$ and so, $a = 0$

The only solution is $a = 0, b = 0, c = 0$. Hence, given vectors are L.I.

Alter method: Construct a matrix A, whose columns are given by vectors.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}_{4\times 3} \quad \begin{array}{l} \text{[Constant coeff.]} \\ \text{[Coeff. of } x\text{]} \\ \text{[Coeff. of } x^2\text{]} \\ \text{[Coeff. of } x^3\text{]} \end{array}$$

Operating $R_2 \to R_2 + R_1, R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Operating $R_4 \leftrightarrow R_3$,

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This is row-reduced echelon form of matrix A.

$$\rho(A) = 3 = \text{no. of columns of } A$$

Hence, given vectors are L.I.

Example 4.7. *Determine whether the given set of vectors (1, –5, –2, 3), (1, 0, 0, –1), (1, 0, 2, 4) are L.D. or L.I.*

Solution. Let $u = (1, -5, -2, 3)$, $v = (1, 0, 0, -1)$, $w = (1, 0, 2, 4)$.

Let $au + bv + cw = 0$ for some scalars $a, b, c \in \mathbf{R}$

$\therefore \quad a(1, -5, -2, 3) + b(1, 0, 0, -1) + c(1, 0, 2, 4) = 0$

$\Rightarrow \quad (a + b + c, -5a, -2a + 2c, 3a - b + 4c) = (0, 0, 0, 0)$

Equating the corresponding elements, we get

$$a + b + c = 0 \quad \text{...(1)}$$

$$-5a = 0 \quad \Rightarrow \quad a = 0$$

$$-2a + 2c = 0 \quad \text{...(2)}$$

$$3a - b + 4c = 0 \quad \text{...(3)}$$

From (2), we get $c = 0$

Using $a = 0$ and $c = 0$ in (1), we get

$$b = 0$$

$\therefore \quad a = 0, b = 0, c = 0$ is the only solution.

Hence, the given set of vectors are L.I.

EXERCISE 4.2

1. Determine which of the following sets of vectors ar L.D. or L.I.
   i. $\{(2, 3, 1), (-1, 4, -2), (1, 18, -4)\}$ ii. $\{(0, 2, -4), (1, -2, -1), (1, -4, 3)\}$
   iii. $\{(1, 2, 3), (0, 1, 2), (-1, 4, 5)\}$
   iv. $\{(2, 3, -1, -1), (1, -1, -2, -4), (3, 1, 3, -2), (6, 3, 0, -7)\}$
2. Find p if the vectors $(1, -1, 3)$, $(1, p, 3)$ and $(1, 0, 1)$ are L.D.
3. Find k if the vectors $(2, 0, k)$, $(3, -1, 5)$, $(5, -1, 1)$ are L.D.
4. In the vector space of polynomial of degree ≤ 4, which of the following sets are L.I.?
   i. $x + 1, x^3 + x^2, x + x^2, x^3 + x^4, x^4 - 1$ ii. $x^3 + 1, x^3 - 1, x, x^4 - x$

Answers

1. i. L.D. ii. L.D. iii. L.I. iv. L.D.
2. $p = -1$ **3.** $k = -4$ **4.** i. L.D. ii. L.I.

4.3 LINEAR COMBINATION OF VECTORS

Let $V(F)$ be a vector space. A vector $v \in V$ is said to be linear combination (L.C.) of the vectors $v_1, v_2, ..., v_n \in V$ if v can be written as

$$v = a_1v_1 + a_2v_2 + ... + a_nv_n$$

where a_i's are scalar $\in F$.

Illustration:

i. Let $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, $v_3 = (0, 0, 1)$, then

$$V = (2, 3, 4)$$

is a linear combination of the vectors v_1, v_2 and v_3 and can be expressed as

$$(2, 3, 4) = 2(1, 0, 0) + 3(0, 1, 0) + 4(0, 0, 1)$$

$$V = 2v_1 + 3v_2 + 4v_3$$

ii. Zero vector **0** can also be expressed as L.C. of finite number of vectors $v_1, v_2 \ldots v_n$ as

$$\mathbf{0} = 0v_1 + 0v_2 + \ldots + 0v_n$$

SOME SOLVED EXAMPLES

Example 4.8. *Write the vector $u = (2, -5, 4)$ as L.C. of vectors $v_1 = (1, -3, 2)$ and $v_2 = (2, -1, 1)$ in vector space $V_3(\mathbf{R})$.*

Solution. Let $u = av_1 + bv_2$; $a, b \in \mathbf{R}$

$$\Rightarrow \quad (2, -5, 4) = a(1, -3, 2) + b(2, -1, 1)$$

$$(2, -5, 4) = (a + 2b, -3a - b, 2a + b)$$

Equating the corresponding elements, we get

$$a + 2b = 2 \quad \ldots(1)$$

$$-3a - b = -5 \quad \ldots(2)$$

$$2a + b = 4 \quad \ldots(3)$$

On adding (2) and (3), we get

$$a = 1$$

From (3), we get $\quad b = 2$

But $a = 1, b = 2$ does not satisfy (1)

as $\quad a + 2b = 1 + 4 = 5 \neq 2$

Hence, given vector cannot be expressed as L.C. of v_1 and v_2.

Example 4.9. *Find the condition on a, b, c such that the matrix $\begin{bmatrix} a & -b \\ b & c \end{bmatrix}$ is a L.C. of $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.*

Solution. Let

$$\begin{bmatrix} a & -b \\ b & c \end{bmatrix} = a_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \ldots(1)$$

where $a_1, a_2, a_3 \in \mathbf{R}$

$$\Rightarrow \quad \begin{bmatrix} a & -b \\ b & c \end{bmatrix} = \begin{bmatrix} a_1 + a_2 + a_3 & a_1 + a_2 - a_3 \\ -a_2 & -a_1 \end{bmatrix}$$

By the definition of equality of two matrices, we have

$$a = a_1 + a_2 + a_3 \quad \ldots(2)$$

$$-b = a_1 + a_2 - a_3 \quad \ldots(3)$$

$$b = -a_2 \qquad \text{...(4)}$$

$$c = -a_1 \qquad \text{...(5)}$$

On adding (2) and (3), we get

$$a - b = 2a_1 + 2a_2 \qquad \text{...(6)}$$

Using (4) and (5) in (6), we get

$$a - b = -2c - 2b$$

$$a + b + 2c = 0$$

which is the required condition.

Alter method. Construct a matrix $[A : b]$ by writing the elements of matrix in columns *i.e.*,

$$[A : b] = \begin{bmatrix} 1 & 1 & 1 & : & a \\ 1 & 1 & -1 & : & -b \\ 0 & -1 & 0 & : & b \\ -1 & 0 & 0 & : & c \end{bmatrix}$$

Operate $R_2 \to R_2 - R_1, R_4 \to R_4 + R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & a \\ 0 & 0 & -2 & : & -b-a \\ 0 & -1 & 0 & : & b \\ 0 & 1 & 1 & : & c+a \end{bmatrix}$$

Operate $R_4 \to R_4 + R_3, R_2 \to \dfrac{-R_2}{2}$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & a \\ 0 & 0 & 1 & : & \dfrac{b+a}{2} \\ 0 & -1 & 0 & : & b \\ 0 & 0 & 1 & : & c+a+b \end{bmatrix}$$

Operate $R_3 \to (-1)R_3, R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & a \\ 0 & 0 & 1 & : & \dfrac{a+b}{2} \\ 0 & 1 & 0 & : & -b \\ 0 & 0 & 0 & : & \dfrac{a+b+2c}{2} \end{bmatrix}$$

Operate $R_3 \leftrightarrow R_2$,

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & a \\ 0 & 1 & 0 & : & -b \\ 0 & 0 & 1 & : & \dfrac{a+b}{2} \\ 0 & 0 & 0 & : & \dfrac{a+b+2c}{2} \end{bmatrix}$$

Required condition is $a + b + 2c = 0$.

Example 4.10. *Find k so that $w = (1, k, 4)$ is a L.C. of $u = (1, 2, 3)$ and $v = (2, 3, 1)$.*

Solution. Since w is L.C. of u and v, then there exists scalars $a_1, a_2, \in \mathbf{R}$ such that

$$w = a_1 u + a_2 v$$

$$(1, k, 4) = a_1(1, 2, 3) + a_2(2, 3, 1)$$

$$= (a_1 + 2a_2, 2a_1 + 3a_2, 3a_1 + a_2)$$

Equating the corresponding elements, we have

$$a_1 + 2a_2 = 1 \quad \text{...(1)}$$

$$2a_1 + 3a_2 = k \quad \text{...(2)}$$

$$3a_1 + a_2 = 4 \quad \text{...(3)}$$

From (1) and (3), we get

$$a_1 = \frac{7}{5}, \qquad a_2 = \frac{-1}{5}$$

Substituting these values in (2), we get

$$\frac{14}{5} - \frac{3}{5} = k \qquad \Rightarrow \qquad k = \frac{11}{5}$$

Alter method: Construct a matrix A by writing u and v in columns *i.e.*,

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ k \\ 4 \end{bmatrix}$$

So $$[A : B] = \begin{bmatrix} 1 & 2 & : & 1 \\ 2 & 3 & : & k \\ 3 & 1 & : & 4 \end{bmatrix}$$

Operate $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & : & 1 \\ 0 & -1 & : & k-2 \\ 0 & -5 & : & 1 \end{bmatrix}$$

Operate $R_3 \to R_3 - 5R_2$, $$\sim \begin{bmatrix} 1 & 2 & : & 1 \\ 0 & -1 & : & k-2 \\ 0 & 0 & : & 11-5k \end{bmatrix}$$

Since w is L.C. of u and v, so

$$11 - 5k = 0$$

$$\Rightarrow \qquad k = \frac{11}{5}.$$

Example 4.11. *Can the polynomial $3x^2 - 5x + 7$ be expressed as L.C. of the polynomials $2x^2 + 7x - 3$ and $x^2 + 3x - 5$.*

Solution. Let $3x^2 - 5x + 7 = a(2x^2 + 7x - 3) + b(x^2 + 3x - 5)$ where $a, b \in \mathbf{R}$.

$$3x^2 - 5x + 7 = (2a + b)x^2 + (7a + 3b)x + (-3a - 5b)$$

Comparing the coeff. of like power of x, we get

$$3 = 2a + b \quad \text{...(1)}$$

$$-5 = 7a + 3b \qquad ...(2)$$

$$7 = -3a - 5b \qquad ...(3)$$

From (1) and (2), we get, $a = -14, b = 31$

Substituting these values in (3),

$$-3(-14) -5(31) = 42 - 155 = -113 \neq 7 \text{ (R.H.S. of eqn. (3))}$$

Hence, eq. (1), (2) and (3) has no solution.

Thus, $3x^2 - 5x + 7$ cannot be expressed as L.C. of $2x^2 + 7x - 3$ and $x^2 + 3x - 5$.

Alter method: Construct matrix A by writing the coeff. of different power of x in columns *i.e.*,

$$A = \begin{bmatrix} -3 & -5 \\ 7 & 3 \\ 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 7 \\ -5 \\ 3 \end{bmatrix}$$

$$[A : B] = \begin{bmatrix} -3 & -5 & : & 7 \\ 7 & 3 & : & -5 \\ 2 & 1 & : & 3 \end{bmatrix}$$

Operate $R_2 \to R_2 + \frac{7}{3} R_1, R_3 \to R_3 + \frac{2}{3} R_1$

$$\sim \begin{bmatrix} -3 & -5 & : & 7 \\ 0 & -26/3 & : & 34/3 \\ 0 & -7/3 & : & 25/3 \end{bmatrix}$$

Operate $R_2 \to 3R_2, R_3 \to 3R_3$

$$\sim \begin{bmatrix} -3 & -5 & : & 7 \\ 0 & -26 & : & 34 \\ 0 & -7 & : & 25 \end{bmatrix}$$

Operate $R_3 \to R_3 - \frac{7}{26} R_2$

$$\sim \begin{bmatrix} -3 & -5 & : & 7 \\ 0 & -26 & : & 34 \\ 0 & 0 & : & 206/13 \end{bmatrix}$$

$\rho(A) = 2, \rho(A : B) = 3$, system has no solution.

Hence $3x^2 - 5x + 7$ can't be expressed as L.C. of given polynomials.

4.3.1 Linear Span

Let $V(F)$ be a vector space and $S = \{v_1, v_2, ..., v_n\}$ be a non-empty subset of vector space $V(F)$, then the set of all L.C. of finite set of elements of S is linear span. It is denoted by $L(S)$.

$$L(S) = < S >$$

$$= \left\{ \sum_{i=1}^{n} a_i v_i \; ; a_i \in F, v_i \in S, n \text{ finite} \right.$$

i.e., every vector of V is a L.C. of vectors in S.

For example: The set {(1, 0, 0), (0, 1, 0), (0, 0, 1)} is spanning set of $\mathbf{R}^3(\mathbf{R})$ as every vector of $\mathbf{R}^3$ can be expressed as L.C. of three vectors.

$$(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1).$$

SOME SOLVED EXAMPLES

Example 4.12. *Examine whether (1, –3, 5) belong to the vector space generated by S = {(1, 2, 1), (1, 1, –1), (4, 5, –2)} or not.*

Solution. Given set is $S = \{(1, 2, 1), (1, 1, -1), (4, 5, -2)\}$

Let $a, b, c \in \mathbf{R}$ are scalar, we have to check that (1, –3, 5) can be expressed as L.C. of vectors of S or not.

$$(1, -3, 5) = a(1, 2, 1) + b(1, 1, -1) + c(4, 5, -2)$$

$$(1, -3, 5) = (a + b + 4c, 2a + b + 5c, a - b - 2c)$$

Equating corresponding elements, we get

$$1 = a + b + 4c \qquad ...(1)$$

$$-3 = 2a + b + 5c \qquad ...(2)$$

$$5 = a - b - 2c \qquad ...(3)$$

On adding (1) and (3), we get

$$2a + 2c = 6 \quad \Rightarrow \quad a + c = 3 \qquad ...(4)$$

On subtracting (2) from (1), we get

$$-a - c = 4 \quad \Rightarrow \quad a + c = -4 \qquad ...(5)$$

Eq. (4) and (5) has no solution.

Thus, given vector (1, –3, 5) cannot be expressed as L.C. of vectors of S.

Example 4.13. *Show that the S = {(0, 1, 1), (1, 0, 1), (1, 1, 0)} spans* $\mathbf{R}^3$ *and write the vector (2, 4, 8) as a linear combination of vectors in S.*

Solution. Here $S = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$.

Let $(x, y, z) \in \mathbf{R}^3$ be any arbitrary vector.

Now express (x, y, z) as a L.C. of vectors of S.

There exist scalars $a, b, c \in \mathbf{R}$ such that

$$a(0, 1, 1) + b(1, 0, 1) + c(1, 1, 0) = (x, y, z)$$

$$(b + c, a + c, a + b) = (x, y, z)$$

On equating the corresponding elements, we have

$$b + c = x \qquad ...(1)$$

$$a + c = y \qquad ...(2)$$

$$a + b = z \qquad ...(3)$$

Subtracting (1) from (2), we get

$$a - b = y - x \qquad ...(4)$$

On adding (3) and (4), we get

$$a = \frac{y - x + z}{2}$$

From (4),
$$b = a - y + x = \frac{y - x + z}{2} - y + x = \frac{x - y + z}{2} \quad ...(5)$$

From (1), we get,
$$c = x - b = x - \frac{x - y + z}{2} \quad \text{[using (5)]}$$
$$= \frac{x + y - z}{2}$$

Thus,
$$(x, y, z) = \frac{y - x + z}{2}(0, 1, 1) + \frac{x - y + z}{2}(1, 0, 1) + \frac{x + y - z}{2}(1, 1, 0)$$

Thus, set S spans $\mathbf{R}^3$ *i.e.*, every vector in $\mathbf{R}^3$ can be expressed as L.C. of vectors of S.

Now, $(x, y, z) = (2, 4, 8) \Rightarrow x = 2, y = 4, z = 8$
$$(2, 4, 8) = \frac{4 - 2 + 8}{2}(0, 1, 1) + \frac{2 - 4 + 8}{2}(1, 0, 1) + \frac{2 + 4 - 8}{2}(1, 1, 0)$$
$$(2, 4, 8) = 5(0, 1, 1) + 3(1, 0, 1) - 1(1, 1, 0)$$

Thus, (2, 4, 8) is expressed as L.C. of vectors in S.

EXERCISE 4.3

1. Express the vector $v = (4, -5, 9, -7)$ as a L.C. of vectors $v_1 = (1, 1, -2, 1)$, $v_2 = (3, 0, 4 -1)$, $v_3 = (-1, 2, 5, 2)$.
2. Consider the vector $v = (1, -2, k)$ in $\mathbf{R}^3(\mathbf{R})$. For what value of k (if any) the vector v can be expressed as L.C. of vectors $v_1 = (3, 0, -2)$ and $v_2 = (2, -1, -5)$?
3. Express the matrix $v = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ in the vector space of 2 × 2 matrices as a L.C. of $v_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.
4. Which of the following vectors can be written as a L.C. of $<(1, 2, 1), (1, 1, -1), (4, 5, -2)>$?
   i. $(2, -1, -8)$
   ii. $\left(\frac{-1}{3}, \frac{1}{3}, \frac{1}{3}\right)$
5. Which of the following polynomials belong to the vector space spanned/generated by $\{x^3, x^2 + 2x, x^2 + 2, 1 - x\}$?
   i. $3x + 2$
   ii. $3x^2 + x + 5$
   iii. $2x^3 + 3x^2 + 3x + 7$
   iv. $x^4 + 7x + 2$

Answers

1. $v = -3v_1 + 2v_2 - v_3$
2. $k = -8$
3. $v = 2v_1 - v_2 + 2v_3$
4. $(2, -1, -8)$
5. $3x^2 + x + 5$

4.3.2 Basis of a Vector Space

Let $V(F)$ be a vector space. A set of vectors $S = \{v_1, v_2, \ldots v_3\} \in V$ is called a basis of V if

i. $v_1, v_2, \ldots v_n$ are L.I.

ii. $v_1, v_2, \ldots v_n$ space V *i.e.*, every vector of V is a L.C. of the vectors of S.

Remarks:

i. The no. of vectors in a basis is unique but basis of a vectors are not unique.
Example: $\{(1, 0), (0, 1)\}$ is basis of $\mathbf{R}^2(\mathbf{R})$ and also $\{(1, 1), (1, 0)\}$ is basis of $\mathbf{R}^2$. No. of vectors in basis = 2 but basis for $\mathbf{R}^2$ can be infinite.

ii. Basis of a vector space are L.I. sets but L.I. sets of vectors need not be basis of vector space.
Example: $\{(1, 0)\}$ is a L.I. set but not basis of $\mathbf{R}^2$ and $\{(1, 0), (0, 1)\}$ is basis of $\mathbf{R}^2$ and also these vectors are L.I.

4.3.3 Dimension of Vector Space

The number of elements/vectors in any basis of vector space $V(F)$ is called dimension of V and is denoted by dim V.

If dim $V = n$, then V is called *n-dimensional vector space.*

A vector space of finite dimension is called finite dimensional vector space.

Example: $\{(1, 0), (0, 1)\}$ are basis of $\mathbf{R}^2$. $\quad\therefore\quad$ dim $\mathbf{R}^2 = 2$

Results:

i. A set of vectors $v_1, v_2, \ldots v_n \in V$ is basis of V iff each element of V can be uniquely expressed as L.C. of the vectors $v_1, v_2, \ldots v_n$.

ii. If V is a finitely generated vector space, then any two basis of V have same number of elements.

iii. If dim $V = n$ and $S = \{v_1, v_2, \ldots v_n\}$ is L.I. subset of V, then S is a basis of V.

iv. If dim $V = n$ and $S = \{v_1, v_2, \ldots v_n\}$ generates V, then S is a basis of V.

v. If dim $V = n$ and $S = \{v_1, v_2, \ldots v_n, v_{n+1}\}$ having $(n + 1)$ vectors, then S is L.D.
OR A set S contains more vector than dimension is always L.D.

vi. dim (zero vector) = 0 and basis of zero vector = { } or ϕ

Vector space	Standard basis	Dimension
$C^n(C)$	$\{(1, 0, \ldots 0), (0, 1, \ldots 0), \ldots, (0, 0, \ldots 1)\}$	n
$C^n(R)$	$\{(1, 0, \ldots 0), (0, 1, \ldots 0), \ldots, (0, 0, \ldots 1), (i, 0, \ldots 0), (0, i, \ldots 0), \ldots, (0, 0, \ldots, i\}$	$2n$
$R^n(R)$	$\{(1, 0, \ldots 0), (0, 1, \ldots 0), \ldots, (0, 0, \ldots 1)$	n
$P_n(C)$ over C	$\{1, x, x^2 \ldots, x^n\}$	$n + 1$
$P_n(C)$ over R	$\{(1, x, x^2, \ldots, x^n, i, ix, ix^2, \ldots, ix^n\}$	$2(n + 1)$
$P_n(R)$ over R	$\{1, x, x^2, \ldots, x^n\}$	$n + 1$

Vector space	**Standard basis**	**Dimension**
$M_{m,\,n}(C)$ over C	$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix},$ $\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$	mn
$M_{m,\,n}(C)$ over R	$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 \end{bmatrix},$ $\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix},$ $\begin{bmatrix} i & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \begin{bmatrix} 0 & i & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & \dots & i \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 \end{bmatrix},$ $\begin{bmatrix} 0 & 0 & \dots & i \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ i & 0 & \dots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & i & \dots & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & i \end{bmatrix}$	$2(mn)$
$M_n(R)$ over R	$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 \end{bmatrix},$ $\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$	mn

SOME SOLVED EXAMPLES

Example 4.14. *Show that the vectors (2, –1, 0), (3, 5, 1), (1, 1, 2) form a basis of* $\mathbf{R}^3$.

Solution. We know that dim ($\mathbf{R}^3$) = 3 and $S = \{(2, -1, 0), (3, 5, 1), (1, 1, 2)\}$ is a set of three vectors then for S to be basis of $\mathbf{R}^3$ it is sufficient to show that vectors of S are L.I.

Let $a, b, c \in \mathbf{R}$ such that

$$a(2, -1, 0) + b(3, 5, 1) + c(1, 1, 2) = 0$$

$$\Rightarrow (2a + 3b + c, -a + 5b + c, b + 2c) = 0$$

$$\Rightarrow \quad 2a + 3b + c = 0$$
$$-a + 5b + c = 0$$
$$b + 2c = 0$$

On solving these equations, we get

$$a = 0, b = 0, c = 0$$

Hence, S is L.I. and forms basis of $\mathbf{R}^3$.

Alter method: Construct matrix A by writing the vector in columns *i.e.*,

$$A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 5 & 1 \\ 0 & 1 & 2 \end{bmatrix}_{3\times 3}$$

Operate $R_2 \leftrightarrow R_1$ $\sim \begin{bmatrix} -1 & 5 & 1 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Operate $R_2 \to R_2 + 2R_1$ $\sim \begin{bmatrix} -1 & 5 & 1 \\ 0 & 13 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

Operate $R_2 \to R_2 - 13R_3$ $\sim \begin{bmatrix} -1 & 5 & 1 \\ 0 & 0 & -23 \\ 0 & 1 & 2 \end{bmatrix}$

Operate $R_2 \leftrightarrow R_3$ $\sim \begin{bmatrix} -1 & 5 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -23 \end{bmatrix}$

which is row echelon form of A. So, $\rho(A) = 3$ = no. of columns

Hence, given vectors are L.I. and form basis of $\mathbf{R}^3$.

Also dim ($\mathbf{R}^3$) = 3.

Example 4.15. *Determine a basis of space spanned by vectors (–3, 1, 2), (0, 1, 3), (2, 1, 0), (1, 1, 1).*

Solution. Let $S = \{(-3, 1, 2), (0, 1, 3), (2, 1, 0), (1, 1, 1)\}$

Clearly, $\{(-3, 1, 2)\}$ being a *singleton set* of non-zero vector is L.I.

Also, $\{(-3, 1, 2), (0, 1, 3)\}$ is L.I. set of vectors because neither vector is a multiple of other.

Let us now consider the set of three vectors $\{(-3, 1, 2), (0, 1, 3), (2, 1, 0)\}$.

Let $\quad a(-3, 1, 2) + b(0, 1, 3) + c(2, 1, 0) = 0$

$\Rightarrow \qquad (-3a + 2c, a + b + c, 2a + 3b) = 0$

$$\Rightarrow \qquad -3a + 2c = 0 \qquad ...(1)$$
$$a + b + c = 0 \qquad ...(2)$$
$$2a + 3b = 0 \qquad ...(3)$$

Multiply eq. (2) by 2 and subtract it from (1), we get

$$-5a - 2b = 0 \quad \Rightarrow \quad 5a + 2b = 0 \qquad ...(4)$$

On solving (3) and (4), we get

$$a = 0, b = 0$$

Now, from (1), $\qquad c = 0$

Thus, only solution is $a = 0, b = 0, c = 0$.

Thus, the set of vectors $\{(-3, 1, 2), (0, 1, 3), (2, 1, 0)\}$ is a L.I. set.

Also, elements of $S \in \mathbf{R}^3$

$\therefore \quad \dim \mathbf{R}^3 = 3$

Since every set of $(n + 1)$ vectors of an n-dimensional vector space V is L.D.

$\therefore \quad S$ is L.D.

Therefore, $\{(-3, 1, 2), (0, 1, 3), (2, 1, 0)\}$ is L.I. and spans $\mathbf{R}^3$, hence it is basis of $\mathbf{R}^3$.

> **Remark: Extension of Vectors**
> i. Form a matrix A whose rows are given vectors.
> ii. Reduce matrix A into row-echelon form by using elementary row-operation.
> iii. Pick the column which do not have any pivot.
> iv. Extend the set of vector by selecting a vector such that all the column have pivot.

Example 4.16. *Extend the set of vector $\{(1, 2, 3), (2, 2, 3)\}$ to form basis of* $\mathbf{R}^3$.

Solution. Construct a matrix A whose rows are these given vector *i.e.*,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{bmatrix}$$

Operating $R_2 \to R_2 - 2R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \end{bmatrix}$$

which is row-echelon form and third column does not contain the pivot.

$\therefore \quad$ The three vectors $\{(1, 2, 3), (2, 2, 3), (0, 0, 1)\}$ are L.I. and form basis of $\mathbf{R}^3$.

> **Remark:** Extension of basis is not unique.

Example 4.17. *Extend the set of vectors $\{(1, 2, 3), (2, 1, 0)\}$ to form a basis of* $\mathbf{R}^3$.

Solution. As $\qquad \dim \mathbf{R}^3 = 3$

Let $\qquad A = \{(1, 2, 3), (2, 1, 0)\}$

Clearly A is L.I. set because none of them is scalar multiple of other. Now, we need one more vector to form set A a basis of $\mathbf{R}^3$.

Standard basis of $\mathbf{R}^3$ are $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

Let us consider $S = \{(1, 2, 3), (2, 1, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

Start with first three vectors and check whether they form a L.I. set or not.

Let $a(1, 2, 3) + b(2, 1, 0) + c(1, 0, 0) = 0$

$\Rightarrow \quad (a + 2b + c, 2a + b, 3a) = 0$

$\Rightarrow \quad a + 2b + c = 0 \qquad ...(1)$

$2a + b = 0 \qquad ...(2)$

$3a = 0 \quad \Rightarrow \quad a = 0 \qquad ...(3)$

Using (3) in (2), we get $\quad b = 0$

Now, from (1), we get $\quad a = 0$

Thus, only solution of (1), (2) and (3) is $a = 0, b = 0, c = 0$.

$\therefore$ Set $\{(1, 2, 3), (2, 1, 0), (1, 0, 0)\}$ is L.I.

Since $\quad \dim \mathbf{R}^3 = 3$

$\therefore$ Set $\{(1, 2, 3), (2, 1, 0), (1, 0, 0)\}$ forms a basis and it is required extended set.

Alter method. Construct a matrix A by writing the vector in rows *i.e.*,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix}$$

Operating $R_2 \to R_2 - 2R_1$, $\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix}$

Operating $R_2 \to \left(\frac{-1}{3}\right) R_2$, $\sim \begin{bmatrix} \textcircled{1} & 2 & 3 \\ 0 & \textcircled{1} & 2 \end{bmatrix}$

which is row echelon form of A and 3rd column does not contain the pivot.

Hence, the vectors $\{(1, 2, 3), (2, 1, 0), (0, 0, 1)\}$ are L.I. and form a basis of $\mathbf{R}^3$.

Example 4.18. *Determine the basis and dimension of w generated by the vectors (1, –1, 1), (8, 4, 2), (2, 2, 0), (3, 9, –3) of* $\mathbf{R}^3$.

Solution. We have to check that how many vectors are L.I. among these four vectors.

Construct a matrix A by writing the vectors in rows

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 8 & 4 & 2 \\ 2 & 2 & 0 \\ 3 & 9 & -3 \end{bmatrix}$$

Operating $R_2 \to R_2 - 8R_1, R_3 \to R_3 - 2R_1, R_4 \to R_4 - 3R_1$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 12 & -6 \\ 0 & 4 & -2 \\ 0 & 12 & -6 \end{bmatrix}$$

Operating $R_2 \to \frac{1}{6} R_2, R_3 \to \frac{1}{2} R_3, R_4 \to \frac{1}{6} R_4$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

Operating $R_3 \to R_3 - R_2, R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The non-zero rows forms basis of w.

$\therefore$ $\{(1, -1, 1), (0, 2, -1)\}$ forms basis of w.

$\therefore$ $\dim w = 2.$

Alter method. Clearly (1, –1, 1) being a non-zero vector is L.I. and {(1, –1, 1), (8, 4, 2)} are L.I. because none of them is scalar multiple of other.

Now check {(1, –1, 1), (8, 4, 2), (2, 2, 0)} are L.D. or L.I. vectors.

Let $a, b, c \in \mathbf{R}$ are scalars such as $a(1, -1, 1) + b(8, 4, 2) + c(2, 2, 0) = 0$

$$[a + 8b + 2c, -a + 4b + 2c, a + 2b) = 0$$

$$\Rightarrow \quad a + 8b + 2c = 0 \quad ...(1)$$

$$-a + 4b + 2c = 0 \quad ...(2)$$

$$a + 2b = 0 \quad ...(3)$$

Subtracting (2) from (1), we get

$$2a + 4b = 0 \quad \Rightarrow \quad a + 2b = 0$$

$$\Rightarrow \quad a = -2b$$

From (1), $\quad -2b + 8b + 2c = 0$

$$6b + 2c = 0$$

$$c = -3b$$

$\therefore$ $a = -2b, c = -3b$ is a soltuion for every value of b.

In particular, if $b = 1 \Rightarrow a = -2, b = -3$ is a solution.

Hence, {(1, –1, 1), (8, 4, 2), (2, 2, 0)} are L.D.

Similarly check the set {(1, –1, 1), (8, 4, 2), (3, 9, –3)}.

This set is also L.D.

Thus, basis of w = {(1, –1, 1), (8, 4, 2)} and $\dim w = 2$.

EXERCISE 4.4

1. Show that the following sets of vectors are basis of $\mathbf{R}^4$:

i. {(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 0, 1), (0, 0, 1, 0)}

ii. {(1, 0, 0, 0), (1, 2, 0, 0), (1, 2, 3, 0), (1, 2, 0, 4)}

2. Show that the following sets of vectors form a basis of $\mathbf{R}^3$:
   i. {(4, 3, 2), (2, 1, 0), (–1, 1, –1)}   ii. {(–1, 1, 0), (0, 3, –3), (2, 0, 1)}
   iii. {(2, 1, 4), (1, –1, 2), (3, 1, –2)}
3. Determine the basis of the sub-space spanned by the vectors {(1, 1, 1), (1, 0, –1), (3, –1, 0), (2, 1, –2)}.
4. Determine a basis of the subspace spanned by the vectors {(3, 2, 4), (1, 0, 2), (1, –1, –1), (6, 7, 5)}.
5. Extend the following sets of vectors to form a basis of $\mathbf{R}^3$:
   i. {(1, 2, 3), (2, –2, 0)}   ii. {(2, 1, –3), (1, –2, 2)}
   iii. {(0, 1, 2), (2, –1, 4)}
6. Let V be the vector space of $n \times n$ matrices over $\mathbf{R}$ and w be the subspace generated by $\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}$, $\begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix}$, $\begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix}$. Show that $\left\{\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\right\}$ forms a basis and dim $w = 2$.

Answers

3. {(1, 1, 1), (1, 0, –1), (3, –1, 0)}
4. {(3, 2, 4), (1, 0, 2), (1, –1, –1)}
5. i. (1, 2, 3), (2, –2, 0), (1, 0, 0)
   ii. (2, 1, –3), (1, –2, 2), (1, 0, 0)
   iii. (0, 1, 2), (2, –1, 4), (1, 0, 0)

4.4 LINEAR TRANSFORMATIONS

Let U and V be two vector spaces over a same field F, then a map function $T : U \to V$ is called linear transformation or vector space homomorphism if

i. T preserves addition *i.e.*,

$$T(u + v) = T(u) + T(v) \text{ for all } u, v \in U$$

ii. T preserves scalar multiplication *i.e.*,

$$T(au) = a\,T(u) \text{ for all } a \in F, u \in U$$

Remarks:

1. The two property in the definition of L.T. can be summed up as a single property
$$T(au + v) = aT(u) + T(v) \text{ for all } a \in F, u, v \in U$$
2. If $U = V$, then L.T., $T : U \to U$ is called **linear operator**.
3. A linear transformation $T : U \to F$ where F is field of scalars, is called **linear functional.**
4. **Zero transformation.** The zero map $\mathbf{0} : V \to V$ such that $0(v) = 0$ for all $v \in V$.
5. **Identity transformation.** The identity map $I : V \to V$ such that $I(v) = V$ for all $v \in V$.
6. If a mapping defined in a L.T. is not linear, then mapping is not L.T.

SOME SOLVED EXAMPLES

Example 4.19. *Show that the function* $T : R^3 \to R^2$ *defined by* $T(x, y, z) = (3x - y, x + y - 2z)$ *is a linear transformation.*

Solution. Let $u = (x_1, y_1, z_1) \in \mathbf{R}^3$ and $v = (x_2, y_2, z_2) \in \mathbf{R}^3$.

$$\begin{aligned}\therefore \quad T(u+v) &= T(x_1 + x_2, y_1 + y_2, z_1 + z_2)\\ &= (3x_1 + 3x_2 - y_1 - y_2, x_1 + x_2 + y_1 + y_2, -2z_1 - 2z_2)\\ &= \{(3x_1 - y_1) + (3x_2 - y_2), (x_1 + y_1 - 2z_1) + (x_2 + y_2 - 2z_2)\}\\ &= (3x_1 - y_1, x_1 + y_1 - 2z_1) + (3x_2 - y_2, x_2 + y_2 - 2z_2)\\ &= T(u) + T(v)\end{aligned}$$

Also for $a \in \mathbf{R}$ and $u = (x_1, y_1, z_1) \in \mathbf{R}^3$, we have

$$\begin{aligned}T(au) &= T(ax_1, ay_1, az_1)\\ &= (3ax_1, -ay_1, ax_1 + ay_1 - 2az_{1)})\\ &= a(3x_1 - y_1, x_1 + y_1 - 2z_1)\\ &= a\,T(u)\end{aligned}$$

$\therefore$ T is a linear transformation.

Example 4.20. *Show that the function $T : R^2 \to R^3$ defined by $T(x, y) = (x + 1, 2y, x + y)$ is not a linear transformation.*

Solution. Let $u = (x_1, y_1) \in \mathbf{R}^2$ and $v = (x_2, y_2) \in \mathbf{R}^2$ be arbitrary.

$$\begin{aligned}\therefore \quad T(u) &= T(x_1, y_1) = (x_1 + 1, 2y_1, x_1 + y_1)\\ T(v) &= T(x_2, y_2) = (x_2 + 1, 2y_2, x_2 + y_2)\end{aligned}$$

Now, $$\begin{aligned}T(u+v) &= T(x_1 + x_2, y_1 + y_2)\\ &= (x_1 + x_2 + 2, 2y_1 + 2y_2, x_1 + x_2 + y_1 + y_2) \quad ...(1)\end{aligned}$$

Also, $$T(u) + T(v) = (x_1 + x_2 + 2, 2y_1 + 2y_2, x_1 + y_1 + x_2 + y_2) \quad ...(2)$$

From (1) and (2), we have

$$T(u+v) \neq T(u) + T(v)$$

Hence, T is not a linear transformation.

Example 4.21. *Prove that the function $T : V \to P_2(x)$ where V is the vector space of square matrices defined by $T(A) = a + (b + c)\,x + dx^2$ for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$ is a L.T.*

Solution. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in V$

$$\begin{aligned}\therefore \quad T(A+B) &= T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)\\ &= T\left(\begin{bmatrix} a+e & b+f \\ c+g & d+f \end{bmatrix}\right)\\ &= (a + c) + (b + f + c + g)x + (d + h)x^2\\ &= [a + (b + c)x + dx^2] + [e + (f + g)x + hx^2]\\ &= T(A) + T(B)\end{aligned}$$

Also for $\alpha \in \mathbf{R}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$, we have

$$T(\alpha A) = T\left(\begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}\right)$$
$$= \alpha a + (\alpha b + \alpha c)x + \alpha dx^2$$
$$= \alpha[a + (b + c)x + dx^2]$$
$$= \alpha\, T(A)$$

$\therefore$ T is a linear transformation.

To find linear transformation

Let $T : U \to V$ be a L.T.

Requirement i. Basis of U.

ii. Image of basis under T.

Example 4.22. *Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ is a L.T. such that $T(1, 2) = (3, -1, 5)$ and $T(0, 1) = (2, 1, -1)$. Describe the mapping completely.*

Solution. We can see, $u_1 = (1, 2)$ and $u_2 = (0, 1)$ are L.I. and basis of $\mathbf{R}^2$.

Let $u = (x_1, x_2) \in \mathbf{R}^2$ be arbitrary.

So, u can be expressed as L.C. of u_1 and u_2.

$\therefore$ There exists scalars a and b such that

$$u = au_1 + bu_2$$
$$(x_1, x_2) = a(1, 2) + b(0, 1) \qquad ...(1)$$
$$(x_1, x_2) = (a, 2a + b)$$
$$\Rightarrow \quad x_1 = a$$
$$x_2 = 2a + b \quad \Rightarrow \quad b = x_2 - 2x_1$$

Putting the values of a and b in (1), we get

$$(x_1, x_2) = x_1(1, 2) + (x_1 - 2x_1)\,(0, 1)$$

Now, applying T on both sides

$$T(x_1, x_2) = T[(x_1(1, 2) + (x_2 - 2x_1)\,(0, 1)]$$
$$= x_1\, T(1, 2) + (x_2 - 2x_1)\, \mathrm{T}(0, 1) \qquad [\because\ T \text{ is L.T.}$$
$$T(x_1, x_2) = x_1(3, -1, 5) + (x_2 - 2x_1)(2, 1, -1)$$
$$= (3x_1, -x_1, 5x_1) + (2x_2 - 4x_1, x_2 - 2x_1, -x_2 + 2x_1)$$
$$T(x_1, x_2) = (2x_2 - x_1, -3x_1 + x_2, 7x_1 - x_2) \text{ which is the required L.T.}$$

Example 4.23. *Find a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ such that $T(1, 1, 1) = (1, 1)$ and $T(1, -1, 1) = (0, 1)$.*

Solution. Since dim $\mathbf{R}^3 = 3$ and the set $\{(1, 1, 1), (1, -1, 1)\}$ does not form a basis of $\mathbf{R}^3$, so, we shall extend the set of vectors to get basis of $\mathbf{R}^3$.

Standard basis of $\mathbf{R}^3 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

Let $S = \{(1, 1, 1), (1, -1, 1), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [\text{Operating } R_2 \to R_2 - R_1]$$

which is row-echelon form of A

$$\rho(A) = 3 = \text{no. of columns}$$

So, {(1, 1, 1), (1, –1, 1), (0, 0, 1)} are L.I. and form basis of $\mathbf{R}^3$.

Given that, $T(1, 1, 1) = (1, 1)$

$T(1, -1, 1) = (0, 1)$

Assume that, $T(0, 0, 1) = (0, 0)$

Let $u = (x, y, z) \in \mathbf{R}^3$ be arbitrary and can be expressed as L.C. of vectors of basis.

Let $a, b, c \in \mathbf{R}$ are scalars such as

$$u = a(1, 1, 1) + b(1, -1, 1) + c(0, 0, 1)$$

$\Rightarrow$ $$(x, y, z) = (a + b, a - b, a + b + c)$$

$\Rightarrow$ $$x = a + b$$

$$y = a - b$$

$$z = a + b + c$$

On solving these equations, we get

$$a = \frac{x+y}{2},\ b = \frac{x-y}{2},\ c = z - x$$

Putting these value in (1), we get

$$u = \frac{x+y}{2}(1,1,1) + \frac{x-y}{2}(1,-1,1) + z - x(0,0,1)$$

Applying T on both sides, we get

$$T(x, y, z) = \frac{x+y}{2}T(1,1,1) + \frac{x-y}{2}T(1,-1,1) + z - x(0,0,1) \qquad [\because \ T \text{ is L.T.}]$$

$$T(x, y, z) = \frac{x+y}{2}(1,1) + \frac{x-y}{2}(0,1) + (z - x)(0,0)$$

$$= \left(\frac{x+y}{2}, x\right)$$

which is the required L.T.

Remark. Answers may not unique as it depend upon the extended vector and its image chosen.

Example 4.24. *Find the linear transformation* $T : \mathbf{R}^3 \to \mathbf{R}^3$ *such that* $T(x) = AX$ *where* $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$.

Solution. Suppose that $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbf{R}^3$

$$T(x_1, x_2, x_3) = (y_1, y_2, y_3)$$

$\Rightarrow$ $$T(X) = Y$$

Then, $$Y = AX \qquad [\because \ T(X) = AX]$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ 2x_1 + x_2 + x_3 \\ -x_1 + x_2 + 2x_3 \end{bmatrix}$$

$\therefore$
$$y_1 = x_1 + x_3$$
$$y_2 = 2x_1 + x_2 + x_3$$
$$y_3 = -x_1 + x_2 + 2x_3$$

Hence required linear transformation is

$$T(x_1, x_2, x_3) = (x_1 + x_3, 2x_1 + x_2 + x_3, -x_1 + x_2 + 2x_3).$$

EXERCISE 4.5

1. Determine which of the following mappings are linear transformation:

i. $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y, z) = (z, 2x + y)$

ii. $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y, z) = (|x|, 0)$

iii. $T : \mathbf{R}^2 \to \mathbf{R}^3$ defined by $T(x, y) = (x + y, x - y, y)$

iv. $T : \mathbf{R}^3 \to \mathbf{R}$ defined by $T(x, y, z) = x_1^2 + x_2^2 + x_3^3$

v. $T : \mathbf{R}^2 \to \mathbf{R}$ defined by $T(x, y) = |2x - 3y|$

vi. $T : \mathbf{R}^2 \to \mathbf{R}$ defined by $T(x, y) = xy$

vii. $T : \mathbf{R}^2 \to \mathbf{R}^3$ defined by $T(x, y) = (x + 2y, 3x - 5y, y)$

viii. $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x, y) = (x^2, y)$

ix. $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x, y) = (x \cos\theta + y \sin\theta, -x \sin\theta + y \cos\theta)$

2. Let P be a vector space of all polynomials over real numbers. Then the function $T : P \to P$ defined by $T[p(x)] = \dfrac{d}{dx} p(x),\ p(x) \in P$ is a linear operator on P.

3. Find a linear transformation in each of the following cases:

i. $T : \mathbf{R}^2 \to \mathbf{R}^2$ such that $T(2, 3) = (4, 5)$ and $T(1, 0) = (0, 0)$

ii. $T : \mathbf{R}^3 \to \mathbf{R}^2$ such that $T(1, 1, -1) = (1, 0)$, $T(4, 1, 1) = (0, 1)$ and $T(1, -1, 2) = (1, 1)$

iii. $T : \mathbf{R}^3 \to \mathbf{R}^2$ such that $(1, 1, 1), (-1, 1, -1), (1, 2, 2)$ in $\mathbf{R}^3$ transform to $(1, 1), (1, 1), (1, 0)$ in $\mathbf{R}^2$ under L.T.

4. Find a linear transformation in the following cases:

i. $T : \mathbf{R}^3 \to \mathbf{R}^2$ such that $T(1, 2, 3) = (1, 1)$ and $T(0, 1, 2) = (1, 2)$

ii. $T : \mathbf{R}^3 \to \mathbf{R}^2$ such that $T(1, 1, 0) = (1, 0)$ and $T(1, -1, 0) = (1, 1)$

iii. $T : \mathbf{R}^3 \to \mathbf{R}^2$ such that $T(1, 1, 1) = (1, 0)$ and $T(1, 1, 2) = (1, -1)$

5. Find the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ such that $T(X) = AX$ where the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$.

Answers

1. i. Yes   ii. No   iii. Yes   iv. No
   v. No   vi. No   vii. Yes   viii. No
   ix. Yes

3. i. $T(x, y) = \left(\frac{4y}{3}, \frac{5y}{3}\right)$ ii. $T(x, y, z) = (5x - 12y - 8z, x - 2y - z)$
   iii. $T(x, y, z) = (y, x + y - z)$

4. i. Answer is not unique. Taking last vector to (0, 0), then $T(x, y, z) = (-x + y, -3x + 2y)$
   ii. $T(x, y, z) = \left(x, \frac{x-y}{2}\right)$ iii. $T(x, y, z) = (x, x - z)$

4.5 MATRIX ASSOCIATED WITH LINEAR MAP/TRANSFORMATION

Order Basis: A basis of a vector space is called an ordered basis if its vectors are written in a specific order.

Let $B = \{v_1, v_2, v_3\}$ is a basis of vector space V.

Now by changing the order of vectors let $B' = (v_2, v_1, v_3\}$.

B and B' are same basis but different ordered basis since order of three vector in B and B' is different.

4.5.1 Matrix of a Linear Transformation Relative to Ordered Basis

Let $B = \{u_1, u_2, ..., u_n\}$ and $B' = \{v_1, v_2, ..., v_m\}$ be ordered basis for the finite dimensional vector space U and V respectively. Let $T : U \to V$ be a linear transformation.

Since $T(u_1), T(u_2), \ldots, T(u_n) \in V$ and $\{v_1, v_2, \ldots v_m\}$ is a basis of V, therefore, each $T(u_i)$, $1 \le i \le n$ can be uniquely expressed as a L.C. of vectors $\{v_1, v_2, \ldots, v_m\}$.

Let
$$T(u_1) = a_{11} v_1 + a_{12} v_2 + ... + a_{1m} v_m$$
$$T(u_2) = a_{21} v_1 + a_{22} v_2 + ... + a_{2m} v_m$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$T(u_n) = a_{n1} v_1 + a_{n2} v_2 + ... + a_{nm} v_m,\ a_{ij} \in F \text{ for all } 1 \le i \le n,\ 1 \le j \le m.$$

The coefficient matrix of this system of equation is $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$

The transpose of this matrix is a $m \times n$ matrix, called the matrix of T w.r.t. ordered basis B and B', denoted by $[T : B, B']$ and is given by

$$[T : B, B'] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{bmatrix}_{m \times n}$$

Remarks:

i. The order of matrix of a linear transformation $T : U \to V$ is $m \times n$ *i.e.*, dim $V \times$ dim U.

ii. If $U = V$ and B is the basis used on both sides, then the matrix of T is denoted by $[T : B]$ instead of $[T : B, B']$.

iii. If either B or B' is changed or both are changed, the matrix also changes accordingly.

To find the matrix of a linear transformation $T : R^m \to R^n$ **in standard basis on both sides**, then matrix is formed by picking the first coordinate in defining formula of T and write the coordinate of x, y, z in first row of the matrix and so on.

For example: $T : R^2 \to R^2$ defined by $T(x, y) = (x, -y)$. Let $B = B' = \{(1, 0), (0, 1)\}$ be standard basis of $\mathbf{R^2}$, then $[T : B] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

If we are required to find defining formula of T, when the matrix is $[T : B, B']$, here the procedure: [**When standard basis are given**]. Multiply the scalars in first row by x, y, z etc. and add. This gives the first coordinate of the defining formula. Similarly, other coordinate can be found.

For example: $T : R^2 \to R^3$ whose matrix is $A = \begin{bmatrix} 1 & -1 \\ -2 & 3 \\ 0 & 1 \end{bmatrix}$.

Let $B = \{(1, 0), (0, 1)\}$ and $B' = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ be standard basis of $\mathbf{R^2}$ and $\mathbf{R^3}$ respectively, then $T(x, y) = (x - y, -2x + 3y, y)$.

SOME SOLVED EXAMPLES

Example 4.25. *Find the matrix $[T : B, B']$ of the linear transformation $T : \mathbf{R^3} \to \mathbf{R^2}$ defined by $T(x, y, z) = (x + y, y + z)$ in following cases $B = \{(1, 1, 1), (1, 0, 0), (1, 1, 0)\}$ and $B' = \{(1, 2), (0, 1)\}$.*

Solution. Given linear transformation is

$$T(x, y, z) = (x + y, y + z)$$

$\therefore$ For the ordered basis $B = \{(1, 1, 1), (1, 0, 0), (1, 1, 0)\}$, we have

$$\left.\begin{aligned} T(1, 1, 1) &= (2, 2) \\ T(1, 0, 0) &= (1, 0) \\ T(1, 1, 0) &= (2, 1) \end{aligned}\right\} \quad \text{...(1)}$$

Now, writing $T(1, 1, 1)$, $T(1, 0, 0)$, $T(1, 1, 0)$ as a L.C. of vectors of $B' = \{(1, 2), (0, 1)\}$.

$$T(1, 1, 1) = (2, 2) = a(1, 2) + b(0, 1) \text{ for scalars } a, b \in \mathbf{R} \quad \text{...(2)}$$

We have, $(2, 2) = (a, 2a + b)$

$\Rightarrow$ $a = 2$

and $2a + b = 2 \Rightarrow b = -2$

Putting these value in (2), we get

$$T(1, 1, 1) = (2, 2) = 2(1, 2) + (-2)\ (0, 1)$$

Similarly, $T(1, 0, 0) = (1, 0) = 1(1, 2) + (-2)\ (0, 1)$

$$T(1, 1, 0) = (2, 1) = 2(1, 2) + (-3)\ (0, 1)$$

Matrix T relative to B and B' is transpose of matrix of coefficients in above system is

$$[T : B, B'] = \begin{bmatrix} 2 & 1 & 2 \\ -2 & -2 & -3 \end{bmatrix}$$

Example 4.26. *Find the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ whose matrix $[T : B, B']$ is $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$ where $B = \{(1, 1, 1), (1, 2, 3), (1, 0, 0)\}$ and $B' = \{(1, 1), (1, -1)\}$.*

Solution. Here, $[T : B, B'] = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$

$$T(1, 1, 1) = \text{linear combination of vectors of } B' \text{ using scalars of first column of } [T : B, B']$$
$$= 1(1, 1) + 3(1, -1)$$
$$= (4, -2)$$

Similarly, $T(1, 2, 3) = -1(1, 1) + 1(1, -1) = (0, -2)$

and, $T(1, 0, 0) = 2(1, 1) + 0(1, -1) = (2, 2)$

Now, let $(x, y, z) \in \mathbf{R}^3$ be any vector and

$$(x, y, z) = a(1, 1, 1) + b(1, 2, 3) + c(1, 0, 0) \quad \text{...(1)}$$
$$(x, y, z) = (a + b + c, a + 2b, a + 3b)$$

$\Rightarrow$
$$x = a + b + c$$
$$y = a + 2b$$
$$z = a + 3b$$

On solving these equations, we get

$$a = 3y - 2z, b = z - y, c = x - 2y + z$$

Using these value in (1), we get

$$(x, y, z) = (3y - 2z)\,(1, 1, 1) + (z - y)\,(1, 2, 3) + (x - 2y + z)\,(1, 0, 0)$$

Applying T on both sides,

$$T(x, y, z) = (3y - 2z)\,T(1, 1, 1) + (z - y)\,T(1, 2, 3) + (x - 2y + z)\,T(1, 0, 0)$$

[$\because$ T is L.T.]

$$= (3y - 2z)\,(4, -2) + (z - y)\,(0, -2) + (x - 2y + z)\,(2, 2)$$
$$= (12y - 8z + 2x - 4y + 2z - 6y + 4z - 2z + 2y + 2x - 4y + 2z)$$

So, $T(x, y, z) = (2x + 8y - 6z, 2x - 8y + 4z)$

which is required linear transformation.

Example 4.27. *If the matrix of a linear operator T on $\mathbf{R}^3$ relative to the standard basis is $\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$, then determine the matrix T relative to the basis $B' = \{(1, 2, 2), (1, 1, 2), (1, 2, 1)\}$.*

Solution. Here $[T, B] = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

$\Rightarrow$ $T(1, 0, 0) = 1(1, 0, 0) + (-1)\,(0, 1, 0) + 1(0, 0, 1) = (1, -1, 1)$

$$T(0, 1, 0) = 1(1, 0, 0) + 1(0, 1, 0) + (-1)(0, 0, 1) = (1, 1, -1)$$
$$T(0, 0, 1) = -1(1, 0, 0) + 1(0, 1, 0) + 1(0, 0, 1) = (-1, 1, 1)$$

Let $(a, b, c) \in \mathbf{R}^3$ be arbitrary so (a, b, c) can be expressed as L.C. of vectors $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

We have, $\quad (a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1)$

Applying T on both sides,

$$T(a, b, c) = aT(1, 0, 0) + bT(0, 1, 0) + cT(0, 0, 1) \qquad [\because \ T \text{ is L.T.}]$$
$$\Rightarrow \quad T(a, b, c) = a(1, -1, 1) + b(1, 1, -1) + c(-1, 1, 1)$$
$$= (a + b - c, -a + b + c, a - b + c)$$

which is required linear transformation.

Alter method. Let $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ be standard basis of $\mathbf{R}^3$. Then

$$T(x, y, z) = (x + y - z, -x + y + z, x - y + z)$$

Now, ordered basis $B' = \{(1, 2, 2), (1, 1, 2), (1, 2, 1)\}$, we have

$$T(1, 2, 2) = (1, 3, 1)$$
$$T(1, 1, 2) = (0, 2, 2)$$
$$T(1, 2, 1) = (2, 2, 0)$$

Now, writing $T(1, 2, 2)$, $T(1, 1, 2)$, $T(1, 2, 1)$ as a L.C. of vectors of $B' = \{(1, 2, 2), (1, 1, 2), (1, 2, 1)\}$

$$T(1, 2, 2) = (1, 3, 1) = a(1, 2, 2) + b(1, 1, 2) + c(1, 2, 1) \qquad ...(1)$$
$$(1, 3, 1) = (a + b + c, 2a + b + 2c, 2a + 2b + c)$$
$$\Rightarrow \quad a + b + c = 1$$
$$2a + b + 2c = 3$$
$$2a + 2b + c = 1$$

On solving these equations, we get

$$a = 1, b = -1, c = 1$$

Putting these values in (1), we get

$$T(1, 2, 2) = (1, 3, 1) = 1(1, 2, 2) + (-1)(1, 1, 2) + 1(1, 2, 1)$$

Similarly, $\quad T(1, 1, 2) = (0, 2, 2) = 4(1, 2, 2) + (-2)(1, 1, 2) + (-2)(1, 2, 1)$

and, $\quad T(1, 2, 1) = (2, 2, 0) = -4(1, 2, 2) + 2(1, 1, 2) + 4(1, 2, 1)$

Thus matrix T is relative to basis B' is

$$[T, B'] = \begin{bmatrix} 1 & 4 & -4 \\ -1 & -2 & 2 \\ 1 & -2 & 4 \end{bmatrix}.$$

EXERCISE 4.6

1. Find the matrix $[T : B, B']$ for the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x, y) = (x, -y)$ where

   i. $B = (e_1, e_2)$ and $B' = \{(1, 1), (1, -1)\}$ $\qquad$ ii. $B = \{(1, 1), (1, 0)\}$ and $B' = \{(2, 3), (4, 5)\}$

2. Write the matrix of linear transformation $T : P_3(R) \to P_2(\mathrm{R})$ defined by $T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_3 + (a_2 + a_3)x + (a_0 + a_1)x^2$ relative to the basis $B = \{1, x - 1, (x - 1)^2, (x - 1)^3\}$ and $B' = \{1, x, x^2\}$ respectively.

3. Let V be a vector space of some particular real function and $B = \{1, t, e^t, te^t\}$ is a basis of V. Let $D : V \to V$ be differential operator on V *i.e.*, $D(f) = \dfrac{df}{dt}$. Find the matrix $[D, B]$.

4. Find the matrix representing the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^4$ defined by $T(x, y, z) = (x + y + z, 2x + z, 2y - z, 6y)$ relative to standard basis of $\mathbf{R}^3$ and $\mathbf{R}^4$.

5. Find the matrix representing the transformation $T : \mathbf{R}^2 \to \mathbf{R}^3$ given by $T(x_1, x_2) = (3x_1 - x_2, 2x_1 + 4x_2, 5x_1 - 6x_2)$ relative to the standard basis of $\mathbf{R}^2$ and $\mathbf{R}^3$.

6. i. Find the matrix representing the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ defined by $T(x, y, z) = (x - y + z, 2x + 3y - z/2, x + y - 2z)$ relative to the ordered basis $B = \{e_1, e_2, e_3\}$ and $B' = \{1, 1, 0), (1, 2, 3), (-1, 0, 1)\}$.

ii. Find the matrix representation of the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x, y) = (2y, 3x - y)$ relative to the basis $\{(1, 3), (2, 5)\}$.

7. Find the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^3$ whose matrix is $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 3 \end{bmatrix}$ relative to ordered basis:

$B = \{(1, 1), (-1, 1)\}$ and $B' = \{(1, 1, 1), (1, -1, 1), (0, 0, 1)\}$.

8. Describe the linear operator T on $\mathbf{R}^2$ determined by the matrix $\begin{bmatrix} 1/2 & 1 \\ 2/3 & 4 \end{bmatrix}$, relative to the ordered basis $B = \{(1, 0), (1, 1)\}$.

9. Describe the linear map $T : \mathbf{R}^2 \to \mathbf{R}^3$ determined by matrix $A = \begin{bmatrix} 1 & -1 \\ -2 & 3 \\ 0 & 1 \end{bmatrix}$, relative to the ordered basis: $B = \{(1, 1), (0, 2)\}$ and $B' = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$.

10. Find the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ whose matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ relative to ordered basis:

a. $B = B' = \{e_1, e_2, e_3\}$

b. $B = \{(1, 1, 1), (1, 0, 0), (0, 1, 0)\}$; $B' = \{1, 2, 3), (1, -1, 1), (2, 1, 1)\}$

11. Find the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ w.r.t. the basis $(5, 1, 3), (3, 2, 2), (1, 2, 1)$ when the matrix of transformation is $\begin{bmatrix} 3 & 2 & -2 \\ -1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix}$.

12. If the matrix of a linear transformation T on $\mathbf{R}^3$ relative to the ordered basis $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)$ is $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$ and then find the matrix relative to the basis $B' = \{(0, 1, -1), (-1, 1, 0), (1, -1, 1)\}$.

Answers

1. a. $\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$ b. $\begin{bmatrix} -9/2 & -5/2 \\ 5/2 & 3/2 \end{bmatrix}$ **2.** $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 1 & 0 & -1 & 2 \end{bmatrix}$

3. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **4.** $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 6 & 0 \end{bmatrix}$ **5.** $\begin{bmatrix} 3 & -1 \\ 2 & 4 \\ 5 & -6 \end{bmatrix}$

6. i. $[T : B, B'] = \begin{bmatrix} 2 & 6 & 0 \\ 0 & -3/2 & -1/4 \\ 1 & 11/2 & -5/4 \end{bmatrix}$ ii. $[T : B] = \begin{bmatrix} -30 & -48 \\ 18 & 29 \end{bmatrix}$

7. $T(x, y) = (2y - x, y, -3x + 3y)$ **8.** $T(x, y) = \left(\dfrac{7x+23y}{6}, \dfrac{2x+10y}{3}\right)$

9. $T(x, y) = (-4x + 2y, x, -2x + y)$

10. a. $T(x, y, z) = (x, y, z)$ b. $T(x, y, z) = (x + 2y - 2z, -x + y + 2z, x + y + z)$

12. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

4.6 COMPOSITION (Product) OF TWO LINEAR TRANSFORMATION

Let U, V, W be three vector spaces over a field F. Let $T_1 : U \to V$ and $T_2 : V \to W$ be two linear transformations. We define $T_2T_1 : U \to W$ is called product (composition) of T_2 and T_1 by $(T_2T_1)(u) = T_2(T_1(u))$ for all $u \in V$.

Remarks: i. Composition of two L.T. is again a L.T.

ii. T_2T_1 is defined under the condition that range of $T_1 \subseteq$ domain of T_2.
Or Range of post factor transformation $\subseteq$ domain of pre-factor transformation.

iii. Let U and V be two vector spaces over the field F. If $T_1 : U \to V$ and $T_2 : U \to V$ be two L.T. then

a. Sum of T_1 and T_2 by $T_1 + T_2 : U \to V$ is a L.T. defined by
$$(T_1 + T_2)(u) = T_1(u) + T_2(u) \text{ for all } u \in V.$$

b. If $a \in F$ be any scalar then scalar multiple of T with a is denoted by $aT : U \to V$ is a L.T. and is defined by $(aT)(u) = a(T(u))$ for all $u \in U$.

SOME SOLVED EXAMPLES

Example 4.28. *Let $T_1 : \mathbf{R}^3 \to \mathbf{R}^2$ such that $T_1(x, y, z) = (3x, 4y - z)$ and $T_2 : \mathbf{R}^2 \to \mathbf{R}^2$ such that $T_2(x, y) = (-x, y)$. Compute T_1T_2 and T_2T_1.*

Solution. Range of $T_2(=R^2) \nsubseteq$ Domain of $T_1(=R^3)$

$\therefore$ T_1T_2 is not defined.

Now, T_2T_1 is defined because range of $T_1(=R^2) \subseteq$ Domain of $T_2(=R^2)$.

$$T_2T_1 = \mathbf{R^3} \to \mathbf{R^2} \text{ is a L.T. defined by}$$

$$T_2T_1(x, y, z) = T_2(T_1(x, y, z)) = T_2(T_1(3x, 4y - z) = (-3x, 4y - z)$$

Example 4.29. *Let T and S be two linear transformation from* $\mathbf{R^3} \to \mathbf{R^2}$ *defined by* $T(x, y, z) = (2x - 3y, 7y + 2z)$ *and* $S(x, y, z) = (x - z, y)$. *Compute* $S + T$, $3S$, $2S - 3T$, ST, TS, $T^2(TT)$.

Solution. i. As S and T are L.T. then $S + T$ is also L.T.

$S + T : \mathbf{R^3} \to \mathbf{R^2}$ defined by

$$\begin{aligned}(S + T)(x, y, z) &= S(x, y, z) + T(x, y, z)\\ &= (x - z, y) + (2x - 3y, 7y + 2z)\\ &= (3x - 3y - z, 8y + 2z).\end{aligned}$$

ii. S is L.T., so $3S$ is also L.T.

$3S : \mathbf{R^3} \to \mathbf{R^2}$ defined by

$$\begin{aligned}(3S)(x, y, z) &= 3S(x, y, z) = 3(x - z, y)\\ &= (3x - 3z, 3y)\end{aligned}$$

iii. S and T are L.T., so $2S$, $3T$ are also L.T.

$\therefore$ $2S - 3T$ is L.T.

$2S - 3T : \mathbf{R^3} \to \mathbf{R^2}$ defined by

$$\begin{aligned}(2S - 3T)(x, y, z) &= 2S(x, y, z) - 3T(x, y, z)\\ &= 2(x - z, y) - 3(2x - 3y, 7y + 2z)\\ &= (2x - 2z, 2y) - (6x - 9y, 21y + 6z)\\ &= (-4x + 9y - 2z, -19y - 6z)\end{aligned}$$

iv. Range of $T(=\mathbf{R^2}) \nsubseteq$ domain of $S(=\mathbf{R^3})$

$\therefore$ ST is not defined.

v. Range of $S(=\mathbf{R^2}) \nsubseteq$ domain of $T(=\mathbf{R^3})$

$\therefore$ TS is not defined.

vi. Range of $T(=\mathbf{R^2}) \nsubseteq$ domain of $T(=\mathbf{R^3})$

$\therefore$ T^2 is not defined.

Example 4.30. *Give an example of two L.T.* T_1 *and* T_2 *such that* $T_1T_2 = 0$ *and* $T_2T_1 \neq 0$.

Solution. Let us define $T_1 : \mathbf{R^2} \to \mathbf{R^2}$ by $T_1(x, y) = (0, y)$

and $T_2 : \mathbf{R^2} \to \mathbf{R^2}$ by $T_2(x, y) = (y, 0)$

$\therefore$ T_1T_2 and T_2T_1 both are defined.

Now, $(T_2T_1)(x, y) = T_2(T_1(x, y)) = T_2(0, y) = (y, 0)$

$\Rightarrow$ $T_2T_1 = (y, 0) \neq 0$

Also, $(T_2T_1)(x, y) = T_1(T_2(x, y)) = T_1(y, 0) = (0, 0)$

$\Rightarrow$ $T_1T_2 = 0.$

4.6.1 Inverse of a Linear Transformation (Operator)

A linear transformation $T : U \to V$ is said to be invertible if it is one-one, onto and inverse of T is $T^{-1} : V \to U$ such that $T^{-1}(v) = u$ iff $T(u) = v$.

Remarks: i. $T^{-1} : V \to U$ is also a L.T.

ii. $T_1 : U \to V$ and $T_2 : V \to w$ are two invertible L.T., then T_2T_1 is also invertible and $(T_2T_1)^{-1} = T_1^{-1}\, T_2^{-1}$.

SOME SOLVED EXAMPLES

Example 4.31. *Let T be a linear operator on* $\mathbf{R}^3$ *defined by* $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$. *Show that T is invertible and find* T^{-1}.

Solution. We know that T is invertible iff T is one-one and onto.

***T* is one-one:** Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2) \in \mathbf{R}^3$ be arbitrary vector such that

$$T(u) = T(v)$$

$$\Rightarrow \quad T(x_1, y_1, z_1) = T(x_2, y_2, z_2)$$

$$\Rightarrow \quad (2x_1, 4x_1 - y_1, 2x_1 + 3y_1 - z_1) = (2x_2, 4x_2 - y_2, 2x_2 + 3y_2 - z_2)$$

$$\Rightarrow \quad 2x_1 = 2x_2 \quad \Rightarrow \quad x_1 = x_2$$

$$4x_1 - y_1 = 4x_2 - y_2 \quad \Rightarrow \quad y_1 = y_2$$

$$2x_1 + 3y_1 - z_1 = 2x_2 + 3y_2 - z_2 \quad \Rightarrow \quad z_1 = z_2$$

$$\Rightarrow \quad (x_1, y_1, z_1) = (x_2, y_2, z_2)$$

$$\Rightarrow \quad u = v$$

So T is one-one.

***T* is onto:** Let $(x, y, z) \in \mathbf{R}^3$ be any vector and let $(a, b, c) \in \mathbf{R}^3$ be a vector such that

$$T(a, b, c) = (x, y, z)$$

$$(2a, 4a - b, 2a + 3b - c) = (x, y, z)$$

$$\Rightarrow \quad 2a = x$$

$$4a - b = y$$

$$2a + 3b - c = z$$

On solving these equation, we get

$$a = \frac{x}{2}, b = 2x - y, c = 7x - 3y - z$$

Since $\quad x, y, z \in \mathrm{R} \quad \Rightarrow \quad a, b, c \in \mathbf{R}$

$\therefore \quad (a, b, c) \in \mathbf{R}^3$

Thus T is onto.

$\therefore$ T being one-one and onto is invertible.

Thus, $\quad T(a, b, c) = (x, y, z)$

$$T^{-1}(x, y, z) = (a, b, c)$$

$$T^{-1}(x, y, z) = \left(\frac{x}{2}, 2x - y, 7x - 3y - z\right).$$

Alter method: $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$

Matrix associated with T relative to standard basis is

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & -1 & 0 \\ 2 & 3 & -1 \end{bmatrix}$$

Matrix A is invertible iff $|A| \neq 0$

$$|A| = 2(1-0) - 0 + 0 \qquad \text{[Expanding along } R_1\text{]}$$

$$\Rightarrow \qquad |A| = 2 \neq 0$$

We know,

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{|A|}\begin{bmatrix} 1 & 4 & 14 \\ 0 & -2 & -6 \\ 0 & 0 & -2 \end{bmatrix}'$$

Thus,

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 4 & -2 & 0 \\ 14 & -6 & -2 \end{bmatrix}$$

or

$$A^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 2 & -1 & 0 \\ 7 & -3 & -1 \end{bmatrix}$$

Thus, Linear transformation $T^{-1} : \mathbf{R}^3 \to \mathbf{R}^3$ relative to standard basis is

$$T^{-1}(x, y, z) = \left(\frac{x}{2}, 2x - y, 7x - 3y - z\right)$$

EXERCISE 4.7

1. Let the linear transformation $T_1 : \mathbf{R}^3 \to \mathbf{R}^2$ such that $T_1(x, y, z) = (4x, 3y - 2z)$ and $T_2 : \mathbf{R}^2 \to \mathbf{R}^2$ such that $T_2(x, y) = (-2x, y)$. Compute T_1T_2 and T_2T_1.
2. $T_1 : \mathbf{R}^3 \to \mathbf{R}^2$, $T_2 : \mathbf{R}^3 \to \mathbf{R}^2$ and $T_3 : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformations defined by $T_1(x, y, z) = (y, x + z)$, $T_2(x, y, z) = (2z, x - y)$, $T_3(x, y) = (y, 2x)$.

   Find the defining formula, if exist, for the following linear transformation:

   i. T_1T_3 ii. T_2T_3 iii. T_3T_1 iv. T_3T_2

   v. $T_3T_1 + T_3T_2$
3. Let T_1 and T_2 be linear operators on $\mathbf{R}^2$ defined by $T_1(x, y) = (0, x)$ and $T_2(x, y) = (y, x)$. Compute

   i. $T_1 + T_2$ ii. $2T_2 - 3T_1$ iii. T_2T_1 iv. T_1T_2

   v. T_2^2 vi. T_1^2
4. Let $T : \mathbf{R}^4 \to \mathbf{R}^2$ be given by $T(x, y, z, t) = (x - y + 2z, y - t)$ and let $S : \mathbf{R}^2 \to \mathbf{R}^3$ be given by $S(x, y) = (x + 3y, -x + y, 2y)$. Show that ST is linear transformation.
5. Let T and S be linear operators on $\mathbf{R}^3$ defined by $T(x, y, z) = (x - 3y, -2z, y - 4z)$ and $S(x, y, z) = (2x, 4x - y, 2x + 3y)$. Show that $ST \neq TS$.
6. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ is defined by $T(x, y, z) = (0, x, y)$. Show that $T^2 \neq 0$ but $T^3 = 0$.

7. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ is linear operator defined by $T(x, y, z) = (0, 0, x)$. Show that $T \neq 0$ but $T^2 = 0$.
8. Illustrate with the help of an example that there exist linear transformations $T_1 : \mathbf{R}^2 \to \mathbf{R}^2$ and $T_2 : \mathbf{R}^2 \to \mathbf{R}^2$ such that $T_2T_1 = 0$ but $T_1T_2 \neq 0$.
9. If $T : \mathbf{R}^3 \to \mathbf{R}^3$ be a linear operator defined by $T(x, y, z) = (x + z, x - z, y)$, show that T is a invertible and find T^{-1}.
10. Show that each of the following operators T on $\mathbf{R}^3$ is invertible and find T^{-1}:
    i. $T(x, y, z) = (x - 3y - 2z, y - 4z, z)$
    ii. $T(x, y, z) = (3x, x - y, 2x + y + z)$
11. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear operator defined by $T(x, y) = (y, 2x - y)$. Show that T is invertible and find a formula for T^{-1}.

Answers

1. T_1T_2 is not defined; $T_2T_1(x, y, z) = (-8x, 3y - 2z)$
2. i. not defined
   ii. not defined
   iii. $T_3T_1(x, y, z) = (x + z, 2y)$
   iv. $T_3T_2(x, y, z) = (x - y, 4z)$
   v. $T_3T_1 + T_3T_2(x, y, z) = (2x - y + z, 2y + 4z)$
3. i. $(T_1 + T_2)\,(x, y) = (y, 2x)$
   ii. $(2T_2 - 3T_1)\,(x, y) = (2y, -x)$
   iii. $(T_2T_1)\,(x, y) = (x, 0)$
   iv. $(T_1T_2)\,(x, y) = (0, y)$
   v. $(T_2^{\,2})\,(x, y) = (x, y)$
   vi. $(T_1^{\,2})\,(x, y) = (0, 0)$
8. $T_1(x, y) = (0, x)$, $T_2(x, y) = (x, 0)$
9. $T^{-1}(x, y, z) = \left(\dfrac{x}{2} + \dfrac{y}{2}, z, \dfrac{x}{2} - \dfrac{y}{2}\right)$
10. i. $T^{-1}(x, y, z) = (x + 3y + 14z, y + 4z, z)$
    ii. $T^{-1}(x, y, z) = \left(\dfrac{x}{3}, \dfrac{x}{3} - y, z - x + y\right)$

4.7 NULL SPACE OR KERNEL OF L.T.

Let $T : U \to V$ be a linear transformation. The null space (Kernel) of T is the subset of U consisting of all the vectors whose image under T is $\mathbf{0}$. It is denoted by $N(T)$ or Ker (T)

$$\text{Ker } (T) = N(T) = \{u \in U : T\,(u) = \mathbf{0}\}.$$

4.8 RANGE OR IMAGE OF A LINEAR TRANSFORMATION

Let $T : U \to V$ be a linear transformation. The range space (image) of T is the set of all those vectors of V which are images of vectors of U under T. It is denoted by $R(T)$.

$$R(T) = \{T(u), u \in U\}.$$

4.9 RANK AND NULLITY OF A L.T.

Let $T : U \to V$ be a linear transformation. The rank of T is denoted by $\rho(T)$ and is defined as dimension of $R(T)$.

The nullity of T is denoted by $\mu(T)$ and is defined as dimension of $N(T)$.

4.9.1 Sylvester's Law/Rank–Nullity Theorem

Let $T : U(F) \to V(F)$ is a linear transformation, then

$$\text{Rank } T + \text{Nullity } T = \dim U$$

i.e., $\rho(T) + \mu(T) = \dim U$

or $\dim (R(T)) + \dim (N(T)) = \dim U$

Proof: Let $S = \{u_1, u_2, ..., u_n\}$ be basis of $N(T)$ so that

$$\dim (N(T)) = \mu(T) = n$$

S is basis of $N(T)$, therefore S is L.I. in $N(T)$ and $N(T) \subseteq U$.

$\Rightarrow$ S is L.I. in U and it can be extended to basis $S' = \{u_1, u_2, ..., u_n, u_{n+1}, ..., u_m\}$ for U.

Hence $\dim (N(T)) = \mu(T) = n$

$$\dim U = m$$

We now show that the set

$$P = \{T(u_{n+1}), T(u_{n+2}), ..., T(u_m)\} \text{ is basis of } R(T).$$

Since S' is basis of U and for any vector $u \in U$ can be expressed as *L.C.* of vectors of S'.

$$u = a_1u_1 + a_2u_2 + ... + a_nu_n + a_{n+1}u_{n+1} + ... + a_mu_m$$

Applying T on both sides, we get

$$T(u) = a_1 T(u_1) + a_2 T(u_2) + ... + a_n T(u_n) + a_{n+1} T(u_{n+1}) + ... + a_m T(u_m) \qquad [\because \; T \text{ is a L.T.}]$$

$$T(u) = 0 + 0 + ... + 0 + a_{n+1} T(u_{n+1}) + ... + a_m T(u_m) \qquad [\because \; u_i \in N(T), 1 \le i \le n]$$

$$T(u) = a_{n+1} T(u_{n+1}) + ... + a_m T(u_m)$$

$\therefore$ $T(u)$ is L.C. of vectors of set P and thus, P spans $R(T)$

Now, to show that P is L.T.

$$a_{n+1} T(u_{n+1}) + a_{n+2} T(u_{n+2}) + ... + a_m T(u_m) = \mathbf{0}$$

$$\Rightarrow \quad T(a_{n+1} u_{n+1} + a_{n+2} u_{n+2} + a_m u_m) = \mathbf{0}$$

$$\Rightarrow \quad a_{n+1} u_{n+1} + a_{n+2} u_{n+2} + ... + a_m u_m \in N(T)$$

Since $S = \{u_1, u_2, ..., u_n\}$ basis of $N(T)$

$$\therefore \quad a_{n+1} u_{n+1} + a_{n+2} u_{n+2} + ... + a_m u_m = b_1u_1 + b_2u_2 + ... + b_nu_n$$

$$\Rightarrow \quad (-b_1)u_1 + (-b_2)u_2 + ... + (-b_n)u_n + a_{n+1} u_{n+1} + a_{n+2} u_{n+2} + ... + a_m u_m = 0$$

Since $S' = \{u_1, u_2, ..., u_n, u_{n+1}, ..., u_m\}$ is basis of $U = S'$ is L.I.

$$\therefore \quad -b_1 = -b_2 = ... = -b_n = a_{n+1} = ... = a_m = 0$$

In particular, $a_{n+1} = a_{n+2} = ... = a_m = 0.$

Thus, P is L.I. and spans $R(T)$.

$$\therefore \quad P = \{T(u_{n+1}), T(u_{n+2}), ..., T(u_m)\} \text{ is basis of } R(T)$$

$$\dim (R(T)) = \rho(T) = m - n$$

$$\rho(T) = \dim U - \mu(T)$$

$$\rho(T) + \mu(T) = \dim U$$

which proves the theorem.

SOME SOLVED EXAMPLES

Example 4.32. *For the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^3$ such that $T(x, y) = (x + y, x - y, y)$, find basis and dimension of range space and its null space. Also verify, Rank-nullity theorem.*

Solution. 1. To find null space of T and its dimension; by definition null space

$$= \{u \in \mathbf{R}^2 : T(u) = 0 \in \mathbf{R}\}$$

Let $u = (x, y) \in \mathbf{R}^2$ be an arbitrary elements of null space.

$\Rightarrow$ $T(u) = 0$

$\Rightarrow$ $T(x, y) = 0$

$(x + y, x - y, y) = (0, 0, 0)$

$\Rightarrow$ $x + y = 0$

$x - y = 0$

$y = 0$

On solving these equations, we get

$$x = 0, y = 0$$

Thus, null space is a zero vector.

$$N(T) = \{0\}$$

$\therefore$ Nullity of $T = \dim(N(T)) = 0$.

2. To find range space of T and its dimension:

Let $V \in R(T) \in \mathbf{R}^3$ be an arbitrary element.

Thus, there exist $(x, y) \in \mathbf{R}^2$ such that $V = T(x, y)$

Now, $(x, y) = x(1, 0) + y(0, 1)$

$= xe_1 + ye_2$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$

Applying T on both sides, we get

$T(x, y) = T(xe_1 + ye_2)$

$= x\,T(e_1) + y\,T(e_2)$ [$\because$ T is L.T.]

Now, $T(e_1) = T(1, 0) = (1 + 0, 1 - 0, 0)$

$= (1, 1, 0)$

$T(e_2) = T(0, 1) = (0 + 1, 0 - 1, 1) = (1, -1, 1)$

$\therefore$ $T(x, y) = x(1, 1, 0) + y(1, -1, 1)$

$V = x(1, 1, 0) + y(1, -1, 1)$ ...(1)

Since $V \in R(T)$ is arbitrary, therefore

1. shows that $R(T)$ is spanned by the vectors $S = \{(1, 1, 0), (1, -1, 1)\}$

3. To check S is L.I.

Let $a, b \in F$ are scalars such as

$a(1, 1, 0) + b(1, -1, 1) = 0$

$(a + b, a - b, b) = (0, 0, 0)$

$\Rightarrow$ $a + b = 0$

$a - b = 0$

$b = 0$

On solving these equation, we get

$$a = 0, b = 0$$

Thus, S is L.I. and therefore, it is a basis set of $R(T)$.

$\therefore \quad \dim (R(T)) = 2$

Also, $\dim U = \dim R^2 = 2$

$$\therefore \quad \text{Rank }(T) + \text{Nullity }(T) = 2 + 0 = 2 = \dim R^2$$

Example 4.33. *A linear transformation* $T : \mathbf{R^2} \to \mathbf{R^3}$ *is as* $T(1, 2) = (3, -1, 5)$ *and* $T(0, 1) = (2, 1, -1)$. *Find the defining formula for T and then find range space, rank, null space and nullity.*

Solution. $u_1 = (1, 2)$ and $u_2 = (0, 1)$ are L.I. and basis of R^2.

Let $u = (x, y) \in R^2$ be arbitrary.

So u can be expressed as L.C. of u_1 and u_2.

There exists scalars $a, b \in F$ such that

$$u = au_1 + bu_2$$

$$(x, y) = a(1, 2) + b(0, 1) \qquad \text{...(1)}$$

$$(x, y) = (a, 2a+b)$$

$$\Rightarrow \quad x = a$$

$$y = 2a + b \quad \Rightarrow \quad b = y - 2x$$

Putting the value of a, b in (1), we get

$$(x, y) = x(1, 2) + (y - 2x)(0, 1)$$

Now applying T on both sides,

$$T(x, y) = x\,T(1, 2) + (y - 2x)\,T(0, 1) \qquad [\because \; T \text{ is L.T.}]$$

$$T(x, y) = x(3, -1, 5) + (y - 2x)(2, 1, -1)$$

$$= (-x + 2y, -3x + y, 7x - y)$$

which is the defining formula for T.

Matrix associated with L.T. is

$$A = \begin{bmatrix} -1 & 2 \\ -3 & 1 \\ 7 & -1 \end{bmatrix}$$

Operating $R_2 \to R_2 - 3R_1, R_3 \to R_3 + 7R_1$

$$\sim \begin{bmatrix} -1 & 2 \\ 0 & -5 \\ 0 & 13 \end{bmatrix} \sim \begin{bmatrix} \textcircled{-1} & 2 \\ 0 & \textcircled{-5} \\ 0 & 0 \end{bmatrix} \qquad [\text{Operating } R_3 \to R_3 + 13/5\, R_2]$$

$$R(T) = \text{Range space}$$

$$= \langle(-1, -3, 7), (2, 1, -1)\rangle \subseteq R^3.$$

$$= \text{Basis of } R(T)$$

$$\dim (R(T)) = 2 = \rho(T)$$

Dimension of $N(T)$

Let $u = (x, y) \in N(T) \subseteq R^2$

$$\therefore \quad T(u) = 0$$

$$(-x + 2y, -3x + y, 7x - y) = (0, 0, 0)$$

$$\Rightarrow \qquad -x + 2y = 0$$
$$-3x + y = 0$$
$$7x - y = 0$$

On solving these equation, we get

$$x = 0, y = 0$$

Thus, $\qquad N(T) = \{0\}$

$$\dim (N(T)) = 0 = \mu(T)$$

Also, $\qquad \rho(T) + \mu(T) = 2 + 0 = 2$

$$= \dim R^2$$

Example 4.34. *Find a L.T.* $T : \mathbf{R}^3 \to \mathbf{R}^3$ *whose range space is generated by (1, 0, –1) and (1, 2, 2).*

Solution. Basis of $R(T) = \{(1, 0, -1), (1, 2, 2)\}$

We know that

Basis of R^3 (domain) = $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$

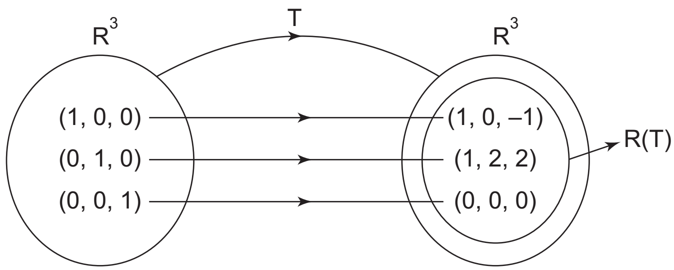


Fig. 4.1

Since $\mathbf{R}^3$ is a three-dimensional vector space and {(1, 0, 0), (0, 1, 0), (0, 0, 1)} is a basis of $\mathbf{R}^3$, there exists a unique L.T., T such that

$$T(1, 0, 0) = (1, 0, -1)$$
$$T(0, 1, 0) = (1, 2, 2)$$
$$T(0, 0, 1) = (0, 0, 0)$$

Let $u = (x, y, z) \in \mathbf{R}^3$ be arbitrary so u can be expressed as L.C. of e_1, e_2, e_3.

$\therefore$ There exists scalars $a, b, c \in F$ such as

$$u = ae_1 + be_2 + ce_3$$
$$(x, y, z) = a(1, 0, 0) + b(1, 0, 0) + c(0, 0, 1) \qquad ...(1)$$
$$(x, y, z) = (a, b, c)$$
$$\Rightarrow \qquad a = x, b = y, c = z$$

Putting these value of a, b and c in (1), we get

$$(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$$

Applying T on both sides, we get

$$T(x, y, z) = xT(1, 0, 0) + yT(0, 1, 0) + zT(0, 0, 1) \qquad [\because \ T \text{ is L.T.}]$$
$$T(x, y, z) = x(1, 0, -1) + y(1, 2, 2) + z(0, 0, 0)$$
$$T(x, y, z) = (x + y, 2y, -x + 2y)$$

which is required L.T.

Example 4.35. *Find a linear transformation* $T : \mathbf{R}^4 \to \mathbf{R}^3$ *whose null space is spanned by (2, 3, 4, 1) and (1, 0, 1, 1).*

Solution. Basis of $N(T) = \{(2, 3, 4, 1), (1, 0, 1, 1)\}$

$$u(T) = \dim (N(T)) = 2$$

$$\dim R^4 = 4$$

So we have to extend the basis of $N(T)$ so that the set becomes basis of R^4.

Standard basis of $R^4 = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$

Let us consider $S = \{(2, 3, 4, 1), (1, 0, 1, 1), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$

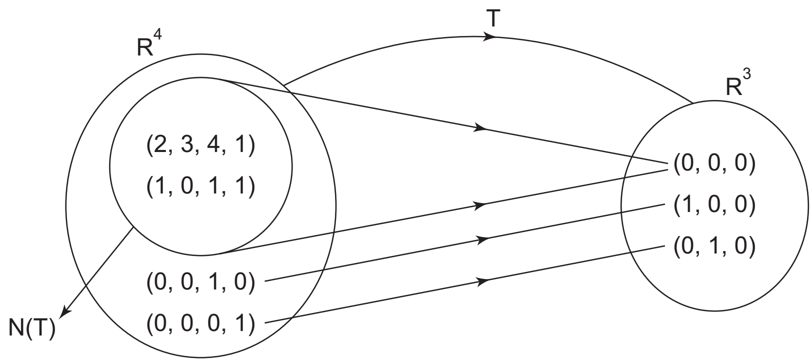


Fig. 4.2

Construct a matrix A by writing 4 vectors in columns *i.e.*,

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Operating $R_1 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 3 & 0 & 0 & 0 \\ 4 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

Operating $R_2 \to R_2 - 3R_1, R_3 \to R_3 - 4R_1, R_4 \to R_4 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -3 & 0 & -3 \\ 0 & -3 & 1 & -3 \\ 0 & -1 & 0 & -2 \end{bmatrix}$$

Operating $R_3 \to R_3 - R_2, R_4 \to R_4 - \frac{R_2}{3} 3R_2$

$$\sim \begin{bmatrix} \textcircled{1} & 1 & 0 & 1 \\ 0 & \textcircled{-3} & 0 & -3 \\ 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & \textcircled{-1} \end{bmatrix}$$

$$\rho(A) = 4 = \text{no. of columns}$$

So, $\{(2, 3, 4, 1), (1, 0, 1, 1), (0, 0, 1, 0), (0, 0, 0, 1)\}$ is basis of $\mathbf{R}^4$.

Now, $T(2, 3, 4, 1) = (0, 0, 0)$ $\quad [\because \quad \in N(T)]$

$T(1, 0, 1, 1) = (0, 0, 0)$

Let $T(0, 0, 1, 0) = (1, 0, 0)$

$T(0, 0, 0, 1) = (0, 1, 0)$

The images of other vectors must be L.I. otherwise these vector belong to $N(T)$ which is not possible as dim $(N(T)) = 2$.

Let $u = (x, y, z, w) \in \mathbf{R}^4$ be arbitrary and can be expressed as L.C. of vectors of S.

There exists scalars $a, b, c, d \in F$ such as

$$u = a(2, 3, 4, 1) + b(1, 0, 1, 1) + c(0, 0, 1, 0) + d(0, 0, 0, 1) \quad ...(1)$$

$\Rightarrow$ $(x, y, z, w) = (2a + b, 3a, 4a + b + c, a + b + d)$

$\Rightarrow$ $x = 2a + b$

$$y = 3a \quad \Rightarrow \quad \frac{y}{3} = a$$

$$z = 4a + b + c$$

$$w = a + b + d$$

On solving, we get $a = \dfrac{y}{3}, b = x - \dfrac{2y}{3}, c = z - x - \dfrac{2y}{3}, d = w - x + \dfrac{y}{3}$

Putting these value in (1), we get

$$(x, y, z, w) = \frac{y}{3}(2,3,4,1) + \left(x - \frac{2y}{3}\right)(1,0,1,1) + \left(z - x - \frac{2y}{3}\right)(0,0,1,0) + \left(w - x + \frac{y}{3}\right)(0,0,0,1)$$

Applying T on both sides, we get

$$T(x, y, z, w) = \frac{y}{3}T(2,3,4,1) + \left(x - \frac{2y}{3}\right)T(1,0,1,1) + \left(z - x - \frac{2y}{3}\right)T(0,0,1,0)$$

$$+\left(w - x + \frac{y}{3}\right)T(0,0,0,1) \quad [\because \quad T \text{ is L.T.}]$$

$$T(x, y, z, w) = \frac{y}{3}(0,0,0) + \left(x - \frac{2y}{3}\right)(0,0,0) + \left(z - x - \frac{2y}{3}\right)(1,0,0) + \left(w - x + \frac{y}{3}\right)(0,1,0)$$

$$T(x, y, z, w) = \left(z - x - \frac{2y}{3}, w - x + \frac{y}{3}, 0\right)$$

which is defining formula of L.T.

EXERCISE 4.8

1. Find R(T), rank (T), $N(T)$ and nullity (T) for the following linear transformations and verify the Sylvester law:
   i. $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y, z) = (x + y, y + z)$
   ii. $T : \mathbf{R}^4 \to \mathbf{R}^3$ defined by $T(x, y, z, w) = (x - y + z + w, x + 2z - w, x + y + 3z - 3w)$

iii. $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x, y) = (x + y, x)$

iv. $T : \mathbf{R}^2 \to \mathbf{R}^3$ defined by $T(x, y) = (x, x + y, y)$

2. Find a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^4$ whose range space is generated by (1, 2, 0, –4) and (2, 0, –1, –3).

3. Find a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ whose range space is generated by (1, 2, 3) and (4, 5, 6).

4. Find a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^4$ whose null space is generated by (0, 1, –3) and (0, –3, 4).

5. Find a linear transformation $T : \mathbf{R}^4 \to \mathbf{R}^3$ whose null space is generated by (1, 2, 3, 4) and (0, 1, 1, 1).

6. Find the range, rank, null space and nullity for zero transformation and the identity transformation on a finite dimension vector space V.

7. Let $T : V \to V$ be a linear map such that $R(T) = N(T)$, where V is finite dimensional. Prove that dim V is even.

8. Let $T : \mathbf{R}^5 \to \mathbf{R}^3$ is a linear transformation such as that $\mu(T) = 2$, then find dim $R(T)$.

9. Is there a linear transformation $T : \mathbf{R}^4 \to \mathbf{R}^2$ such as $\rho(T) = 3$, $\mu(T) = 2$.

10. If $T : \mathbf{R}^4 \to \mathbf{R}^3$ is a linear transformation defined by $T(e_1) = (1, 1, 1)$, $T(e_2) = (1, -1, 1)$, $T(e_3) = (1, 0, 0)$, $T(e_4) = (1, 0, 1)$, verify that $\rho(T) + \mu(T) = \dim \mathbf{R}^4 = 4$.

Answers

1. i. $R(T) = R^2$, $\rho(T) = 2$, $N(T) = \{(1, -1, 1)\}$, $\mu(T) = 1$

ii. $R(T) = \{(1, 1, 1), (0, 1, 2)\}$, $\rho(T) = 2$, $N(T) = \{(2, -1, 1, 0), (1, 2, 0, 1)\}$, $\mu(T) = 2$

iii. $R(T) = R^2$, $\rho(T) = 2$, $N(T) = \{0\}$, $\mu(T) = 0$

iv. $R(T) = \{(1, 1, 0), (0, 1, 1)\}$, $\rho(T) = 2$, $N(T) = \{0\}$, $\mu(T) = 0$

2. $T(x, y, z) = (x + 2y, 2x, -y, -4x - 3y)$ **3.** $T(x, y, z) = (x + 4y, 2x + 5y, 3x + 6y)$

4. $T(x, y, z) = (x, 0, 0, 0)$ **5.** $T(x, y, z, t) = (z - y - x, t - 2x - y, 0)$

7. Zero transformation: Range = {0}, Rank = 0, Null space = V, Nullity = div V

Identity transformation: Range = V, Rank = dim V, Null space = {0}, Nullity = 0

8. 3 **9.** No

INTERESTING FACTS

- A vector space is a collection of mathematical objects called vectors.

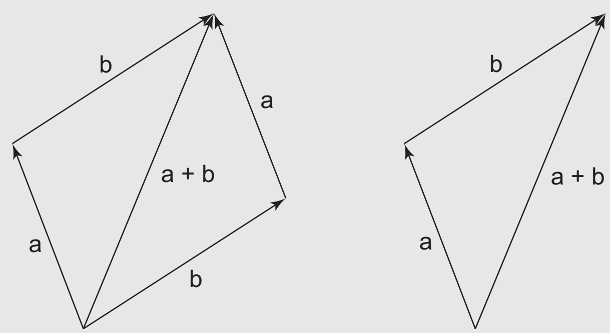


Fig. 4.3

- Two operations are defined in a vector space: addition of two vectors and multiplication of a vector with a scalar. These operations can change the size of a vector and the direction it points to. But the result is still in vector space.
- We cannot change the vector in a way that does not make it, vector anymore.
- Addition of scalar with vector is not possible, because they come under different dimensions in space.

VIDEO REFERENCES

Vector Spaces –Basis and Dimension

Linear Transformations

Invertible Linear Transformations and Matrices

The Matrix of a Linear Transformation

APPLICATIONS TO REAL LIFE

Rank vs L.D. and L.I.

A is located 4 km in north and 4 km in east from my location. Then this tells the unique coordinate of the point *A* from my present location. But in an addition, if I say that point *A* is 5 km in north-east direction, then this is of no use!. It means two informations both are related to different directions are sufficient to locate any point in 2D plane. Similar concept is for higher dimensional planes. In terms of matrix *A* of size $n \times m$, if number of linearly independent vectors are '*r*', it means that with the help of these '*r*' vectors, one can describe all the directions associated with this matrix.

Now the number of linearly independent vectors of matrix is nothing but the rank of matrix. It means for any matrix when you are asked to find the rank, it means in-directly you are giving the sufficient number of directions associated with the matrix.

Relation between Matrix and Linear Transformation

I will try to explain with a camera and a robotic hand. Let's say you have a robotic hand that does something depending upon the input of the camera. Now the camera is placed in the corner of the room. So, from the camera's point of view, distances will be different. Sometimes you need to scale up, scale down or rotate depending upon the orientation. For all these you need linear transformation. You need to transform the matrix from its original form to a different form so that you can work on it.

SUBJECTIVE SOLVED QUESTIONS (HOTS)

Example 1. *Give a basis for each of the following vector space over the indicated fields:*

i. $R(\sqrt{2})$ *over R*   ii. $Q(2^{1/4})$ *over Q*

where Q, R are field of rational and real numbers.

Solution. i. We have $R\,(\sqrt{2}) = (a + \sqrt{2}b : a, b \in R)$

Now zero element of $R(\sqrt{2})$ can be written as $0 = 0 + 0\,.\,\sqrt{2}$

Let $a, b \in R$, such that

$$a.1 + \sqrt{2}b = 0$$

$\Rightarrow$ $$a + \sqrt{2}b = 0 + 0.\sqrt{2}$$

$\Rightarrow$ $$a = 0, b = 0$$

$\Rightarrow$ S is linearly independent.

Let $x + \sqrt{2}y$ be any element of $R(\sqrt{2})$. Then $x + \sqrt{2}y = x \,.\, 1 + \sqrt{2}\,.y$

$\Rightarrow$ every element of $R(\sqrt{2})$ is expressible as the linear combination of elements of S.

$\Rightarrow$ $$L(S) = R(\sqrt{2})$$

Hence, $S = \{1, \sqrt{2}\}$ is a basis of $R(\sqrt{2})$ having two elements so that dim. $R(\sqrt{2}) = 2$

ii. We have

$$Q(2^{1/4}) = \{a + (2^{1/4})b : a, b \in Q\}$$

The zero element of $Q(2^{1/4})$ is $0 = 0 + (2^{1/4})$.

Let $S = \{1, 2^{1/4}\}$, then $S \subseteq Q(2^{1/4})$. Now we shall see that S forms a basis of $Q(2^{1/4})$.

Let $a, b \in Q$ such that $a \,.\, 1 + b \,.\, (2^{1/4}) = 0$

$\Rightarrow$ $$a + b(2^{1/4}) = 0 + 0 \,.\, (2^{1/4})$$

$\Rightarrow$ $$a = 0, b = 0$$

$\Rightarrow$ S is linearly independent.

Let $x + (2^{1/4})y$ be any element of $Q(2^{1/4})$

Then, $$x + (2^{1/4})y = x.1 + (2^{1/4}).y$$

$\Rightarrow$ every element of $Q(2^{1/4})$ is expressible as a linear combination of elements of S.

$\Rightarrow$ $$L(S) = Q(2^{1/4})$$

Hence, $S = \{1, 2^{1/4}\}$ is a basis of $Q(2^{1/4})$ and dim. $Q(2^{1/4}) = 2$

Example 2. *In the space $c[0, \pi]$, let f, g, h and j be the vectors defined by*

$f(x) = 1$, $g(x) = x$, $h(x) = \cos x$, $j(x) = \cos^2 \dfrac{x}{2}$ for $0 \le x \le \pi$. Show that f, g, h and j are linearly dependent by writing j as a linear combination of f, g and h.

Solution. We know that $\cos^2 x = 2\cos^2 x - 1$

Replacing x by $\frac{x}{2}$, we get

$$\cos x = 2\cos^2 \frac{x}{2} - 1$$

$\Rightarrow$ $$\cos x + 1 = 2\cos^2 \frac{x}{2}$$

$\Rightarrow$ $$\frac{\cos x + 1}{2} = \cos^2 \frac{x}{2}$$

or $$\cos^2 \frac{x}{2} = \frac{1}{2}.(\cos x) + \frac{1}{2}.(1)$$

$$j(x) = \frac{1}{2}h(x) + \frac{1}{2}.f(x)$$

or
$$j(x) = \frac{1}{2}.h(x) + 0.g(x) + \frac{1}{2}.f(x)$$

Example 3. *Let T be a linear operator on a vector space V(F). If $T^2 = 0$, what can you say about the relation of the range of T to the null space of T? Give an example of a linear operator on $V_2(R)$ such that $T^2 = 0$ but $T \neq 0$.*

Solution. Since $T^2 = 0$, then for $\alpha \in V$

$$T^2(\alpha) = 0(\alpha) \quad \Rightarrow \quad T[T(\alpha)] = 0$$

$$T(\alpha) \in N(T) \qquad \text{[By definition of null space]}$$

But
$$T(\alpha) \in R(T) \ \forall \ \alpha \in V$$

$$R(T) \subset N(T)$$

Hence when $T^2 = 0$, the range of T is contained in null space of T.

Example 4. *Let V(R) be the vector space of all polynomials in x with coefficients in the field R. Let D and let T be two linear transformations on V defined by*

$$D(f(x)) = \frac{d}{dx} f(x) \ \forall \ f(x) \in V \text{ and } T(f(x)) = xf(x) \ \forall \ f(x) \in V$$

then show that $DT \neq TD$. Also, show that $DT - TD = I$.

Solution. Let $f(x) \in V$. Then

$$(DT)(f(x)) = D[T(f(x))] = D[xf(x)]$$

$$= \frac{d}{dx}[xf(x)] = f(x) + x\frac{d}{dx}f(x) \qquad \text{...(1)}$$

Also,
$$(TD)(f(x)) = T(D(f(x)))$$

$$= T\left(\frac{d}{dx}f(x)\right) = x.\frac{d}{dx}f(x) \qquad \text{...(2)}$$

Therefore, from (1) and (2), we can say that there exists $f(x) \in V$ such that

$$(DT)(f(x)) \neq (TD)(f(x))$$

Hence,
$$DT \neq TD.$$

Also, $(DT)(f(x)) - (TD)(f(x)) = f(x) = I(f(x))$

$\therefore$
$$DT - TD = I$$

Example 5. *Define $T : R^3 \to R^4$ by $T(x) = (x_1 - x_3, x_1 + x_2, x_3 - x_2, x_1 - 2x_2)$ for all $x = (x_1, x_2, x_3) \in$* $\mathbf{R}^3$.

a. *Find T(1, –2, 3)*

b. *Find a vector $x \in R^3$ such that $T(x) = (8, 9, -5, 0)$.*

Solution. $T : \mathbf{R}^3 \to \mathbf{R}^4$ defined by

$$T(x_1, x_2, x_3) = (x_1 - x_3, x_1 + x_2, x_3 - x_2, x_1 - 2x_2)$$

a.
$$T(1, -2, 3) = (1 - 3, 1 + (-2), 3 - (-2), 1 - 2(-2))$$

$$= (-2, -1, 5, 5)$$

b. Let $x = (u, v, w) \in \mathbf{R}^3$ such that

$$T(u, v, w) = (8, 9, -5, 0)$$

$(u - w, u + v, w - v, u - 2v) = (8, 9, -5, 0)$

On comparing, we get

$$u - w = 8 \quad ...(1)$$
$$u + v = 9 \quad ...(2)$$
$$w - v = -5 \quad ...(3)$$
$$u - 2v = 0 \quad \Rightarrow \quad u = 2v \quad ...(4)$$

Subtracting (1) from (2), we get

$$v + w = 1 \quad ...(5)$$

Adding (3) and (5), we get

$$w = -2$$

From (5), $\quad v - 2 = 1 \quad \Rightarrow \quad v = 3$

From (4), $\quad u = 6$

Thus, $(u, v, w) = (6, 3, -2)$ is the required vector in $\mathbf{R}^3$ whose image is $(8, 9, -5, 0)$ in T.

Example 6. *Let $v_1 = (-1, 2, 0)$, $v_2 = (3, 2, -1)$ and $v_3 = (1, 6, -1)$ be three vectors in $\mathbf{R}^3(R)$. Then show that $[v_1, v_2] = [v_1, v_2, v_3]$.*

Solution. Clearly, $\quad [v_1, v_2] = (\lambda_1 v_1 + \lambda_2 v_2 : \lambda_1, \lambda_2 \in R)$

and, $\quad [v_1, v_2, v_3] = \{\mu_1 v_1 + \mu_2 v_2 + \mu_3 v_3 : \mu_1, \mu_2, \mu_3 \in R\}$

Let $\quad v_3 = (1, 6, -1) = \alpha_1 v_1 + \alpha_2 v_2$. Then,

$$(1, 6, -1) = (-\alpha_1 + 3\alpha_2, 2\alpha_1 + 2\alpha_2, -\alpha_2)$$

$\Rightarrow \quad -\alpha_1 + 3\alpha_2 = 1,\ 2\alpha_1 + 2\alpha_2 = 6, -\alpha_2 = -1.$

$\Rightarrow \quad \alpha_1 = 2, \alpha_2 = 1$

$\therefore \quad (1, 6, -1) = 2(-1, 2, 0) + 1(3, 2, -1)$

$\Rightarrow \quad \mu_3(1, 6, -1) = 2\mu_3(-1, 2, 0) + \mu_3(3, 2, -1)$

$\Rightarrow \quad \mu_3 v_3 = 2\mu_3 v_1 + \mu_3 v_2$

Now, $\quad [v_1, v_2, v_3] = \{\mu_1 v_1 + \mu_2 v_2 + 2\mu_3 v_1 + \mu_3 v_2 : \mu_1, \mu_2, \mu_3 \in R\}$

$\Rightarrow \quad [v_1, v_2, v_3] = \{(\mu_1 + 2\mu_3) v_1 + (\mu_2 + \mu_3) v_2 : \mu_1, \mu_2, \mu_3 \in R\}$

$\Rightarrow \quad [v_1, v_2, v_3] = \{\lambda_1 v_1 + \lambda_2 v_2 : \lambda_1, \lambda_2 \in R\}$

$\Rightarrow \quad [v_1, v_2, v_3] = [v_1, v_2]$

Example 7. *Let V be the vector space of 2×2 symmetric matrix over R. Show that dim. $V = 3$.*

Solution. An arbitrary 2×2 symmetric matrix is of the form $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ where $a, b, c \in R$. Now setting i. $a = 1, b = 0, c = 0$, ii. $a = 0, b = 1, c = 0$, iii. $a = 0, b = 0, c = 1$, thus we obtain the following matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We shall show that the set $S = \{A, B, C\}$ is a basis of V.

First we show that S is linearly independent.

Let $x, y, z \in R$ such that $\quad xA + yB + zC = 0$

$$\Rightarrow \quad x\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x & y \\ y & z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad x = 0, y = 0, z = 0$

$\therefore \quad S = \{A, B, C\}$ is linearly independent.

Now, we show that $L(S) = V$.

Let $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ be any element of V, then we have

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & c \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} a & b \\ b & c \end{bmatrix} = aA + bB + cC$$

$\Rightarrow$ Every element of V is linear combination of elements of S.

Example 8. *Let V be a finite dimensional vector space and T be a linear operator on V. Suppose that rank (T^2) = rank (T). Prove that the range and null space of T are disjoint i.e., have only the zero vector in common.*

Solution. We know that, dim. V = rank (T) + nullity (T) ...(1)

Now, T^2 is also a linear operator on V, then

dim. V = rank (T^2) + nullity (T^2) ...(2)

From (1) and (2), we have

rank (T) + nullity (T) = rank (T^2) + nullity (T^2)

$\Rightarrow$ nullity (T) = nullity (T^2) [$\because$ rank (T) = rank (T^2))]

$\Rightarrow$ dim. of null space of T = dim. of null space of T^2.

If $\alpha \in$ null space of T, then

$$T(\alpha) = 0$$

$$\Rightarrow \quad T[T(\alpha)] = T(0)$$

$$\Rightarrow \quad T^2(\alpha) = 0 \qquad [\because \; T(0) = 0]$$

$\therefore \quad \alpha \in$ null space of T^2

$\therefore$ null space of $T \subseteq$ null space of T^2.

But null space of T and null space of T^2 are both sub-spaces of V and have the same dimension.

Then, null space of T = null space of T^2

$\Rightarrow$ null space of $T^2 \subseteq$ null space of T

$$\Rightarrow \quad T^2(\alpha) = 0$$

$$\Rightarrow \quad T(\alpha) = 0$$

Let $\beta \neq 0$ and $\beta \in R(T) \cap N(T)$, then $\beta \in R(T)$ and $\beta \in N(T)$.

Now, $\beta \in N(T) \Rightarrow T(\beta) = 0$

Also $\beta = R(T) \Rightarrow \exists\, \alpha \in V$ such that $T(\alpha) = \beta$

Now $T(\alpha) = \beta$

$\Rightarrow \qquad T[T(\alpha)] = T(\beta) = 0$

Thus there exist $\alpha \in V$ such that $T[T(\alpha)] = 0$ but $T(\alpha) = \beta \neq 0$ which is contradiction to the equation (1). Therefore, there exists no $\beta \in R(T) \cap N(T)$ such that $\beta \neq 0$. Hence $R(T) \cap N(T) = \{0\}$.

SUMMARY

1. For a vector space $V(F)$, 'F' is always a sub-field of V *i.e.*, V can be defined only over its sub-field.
2. If $\rho(A)$ = number of columns, then vectors in 'A' are L.I. otherwise L.D.
3. If dim $V = n$ (finite dimensional) and $S = \{v_1, v_2, ..., v_n\}$ is L.I. subset of V, then S is basis of V.
4. The number of vectors in any basis of $V(F)$ is called dimension of V.
5. dim. of range space = Rank

   i.e., $\quad \dim R(T) = \rho(T)$

   dim. of null space = Nullity

   i.e., $\quad \dim N(T) = \mu(T)$
6. Rank-Nullity theorem states that

   $$\rho(T) + \mu(T) = \text{dim. } U, \text{ where '}U\text{' is a vector space}$$
7. The linear map $T : U(F) \rightarrow V(F)$ is called linear transformation if

   $$T(au + v) = aT(u) + T(v)\ \forall\ a \in F, u, v \in U$$
8. Composition of two linear transformation is defined only when

   Range of Post-Factor transformation $\subseteq$ domain of pre-factor transformation.

OBJECTIVE QUESTIONS

1. Consider the set of all functions f defined on [0, 1], such that

   I. $f\left(\frac{1}{2}\right) = 0$ II. $f\left(\frac{3}{4}\right) = 1$ III. $f(x) = xf(x)$ IV. $f(0) = f(1)$

   Then the correct code is

   a. only (I) is vector space b. only (I) and (IV) are vector spaces

   c. only (I), (III), and (IV) are vector spaces d. All are vector spaces
2. Consider the vector space V over the field of real numbers spanned by the set $S = \{(0, 1, 0, 0), (1, 1, 0, 0), (1, 0, 1, 0), (0, 0, 1, 0), (1, 1, 1, 0), (1, 0, 0, 0)\}$

   What is the dimension of V?

   a. 1 b. 2 c. 3 d. 4
3. Let $X = (3, 2, -1)$, $Y = (2, 4, 1)$, $Z = (4, 0, -3)$ and $W = (10, 4, -5)$ be vectors in R^3, a real vector space. Which one of the following is correct?

   a. $2X + Z = W.Y + Z = W$ b. $2X - Y = Z.Y + 2Z = W$

   c. $X + Z = W.2X + Y = Z$ d. $Y + 2Z = W.X - Y = Z$
4. Given the vector $\alpha = (1, 2, 3)$, $\beta = (3, 1, 0)$, $\gamma = (2, 1, 3)$ and $\delta = (-1, 3, 6)$.

   Consider the following statements:

   I. γ is a linear combination of α and β.

II. δ is a linear combination of α and β, then which of the following statement given above is/are correct?

a. I only b. II only c. Both I and II d. Neither I nor II

5. Which one of the following is not a basis of R^3?

a. $(3, 0, 0), (0, -1, 0), (0, 0, 1/2)$ b. $(0, 0, -3), (1, 2, 3), (1, 2, 1)$

c. $(1, 2, -1), (1, 1, 1), (1, 3, 5)$ d. $(1, 1, 0), (1, 2, 3), (0, 0, 1)$

6. Which among the following is not linear?

a. $F: R^2 \to R^2$ such that $F(x, y) = (2x - y, x)$ b. $F: R^3 \to R^2$ such that $F(x, y, z) = (z, x + y)$

c. $F: R \to R^2$ such that $F(x) = (2x, 3x)$ d. $F: R^3 \to R^2$ such that $F(x, y, z) = (x + 1, y + z)$

7. The two transformation $T: R^3 \to R^2$ and $S: R^3 \to R^2$ defined as $T(x, y, z) = (x + 1, y + z)$ and $S(x, y, z) = (|x|, 0)$, then

a. T and S both are linear b. T and S both are not linear

c. T is linear but S is not linear d. T is not linear but S is linear

8. The unique linear transformation $T: R^2 \to R^2$ such that $T(1, 2) = (2, 3)$ and $T(0, 1) = (1, 4)$, then the rule for T is,

a. $T(x, y) = (y, -5x + 4y)$ b. $T(x, y) = (-5x + 4y, y)$

c. $T(x, y) = (x, -5x + 4y)$ d. $T(x, y) = (-4x + 5y, y)$

9. A linear transformation $T: R^2 \to R^2$ such that $T(3, 1) = (2, -4)$ and $T(1, 1) = (0, 2)$. Then, $T(7, 8)$ is

a. $(-1, 3)$ b. $(-1, 19)$ c. $(2, -3)$ d. $(-3, 2)$

10. Let F be any field and let T be a linear operator on F^2 defined by $T(a, b) = (a + b, a)$, then $T^{-1}(a, b)$ is equal to

a. $(b, a - b)$ b. $(a - b, b)$ c. $(a, a + b)$ d. $(a + b, a - b)$

11. The matrix of the linear transformation T on R^3 (R) defined as $T(x, y, z) = (2y + z, x - 4y, 3x)$ w.r.t. the basis $B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ is

a. $\begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$ b. $\begin{bmatrix} 3 & -6 & 6 \\ 3 & -6 & 5 \\ 3 & -2 & -1 \end{bmatrix}$ c. $\begin{bmatrix} 3 & -6 & 6 \\ -6 & 5 & -2 \\ -1 & 3 & 3 \end{bmatrix}$ d. $\begin{bmatrix} 3 & 6 & 6 \\ 6 & 5 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

12. Let T be a linear transformation from $R^3 \to R^2$ defined by $T(x, y, z) = (x + y, y - z)$. Then the matrix of T w.r.t. ordered basis $\{(1, 1, 1), (1, -1, 0), (0, 1, 0)\}$ and $\{(1, 1), (1, 0)\}$ is

a. $\begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ b. $\begin{bmatrix} 0 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ c. $\begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}$ d. $\begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 1 & 0 \end{bmatrix}$

13. Let T_1 and T_2 be two linear operators on R^3 defined by $T_1(x, y, z) = (x, x + y, x - y - z)$, $T_2(x, y, z) = (x + 2z, y - z, x + y + z)$. Then,

a. T_1 is invertible but not T_2 b. T_2 is invertible but not T_1

c. both T_1 and T_2 are invertible d. neither T_1 nor T_2 is invertible

14. For a linear transformation $T: R^{10} \to R^6$ the Kernel is having dimension 5. Then, the dimension of the range of T is

a. 5 b. 6 c. 4 d. 2

15. If $T : V_2(R) \rightarrow V_3(R)$ defined as $T(a, b) = (a + b, a - b, b)$ is a linear transformation, then nullity of T is

a. 0 b. 1 c. 2 d. 3

16. Consider the following linear transformation from the vector space R^2 into the vector space R^3

$$T(x, y) = (-x - y, 3x + 8y, 9x - 11y),$$

then, the rank and nullity of T are respectively,

a. 2 and 0 b. 1 and 0 c. 1 and 1 d. –1 and 2

17. Let $T : R^3 \rightarrow R^3$ be the linear transformation defined by

$T(x, y, z) = (x + 2y - z, y + z, x - 2z)$, then the dimension of kernel T is

a. 0 b. 1 c. 2 d. 3

18. Choose the correct matching from a, b, c and d for the transformation T_1, T_2 and T_3 (mapping from $R^2 \rightarrow R^3$) as defined in group I with the statements given in group II:

Group-I	Group-II
P. $T_1(x, y) = (x, x, 0)$	1. L.T. of rank 2
Q. $T_2(x, y) = (x, x + y, y)$	2. Not a L.T.
R. $T_3(x, y) = (x, x + 1, y)$	3. L.T. of rank 1

a. P–3, Q–1, R–2 b. P–1, Q–2, R–3 c. P–3, Q–2, R–1 d. P–1, Q–3, R–2

19. If $T : R^4 \rightarrow R^3$ be the linear transformation defined by

$T(x, y, z, w) = (x - y + z + w, x + 2z - w, x + y + 3z - 3w)$, then the dimension of its range is

a. 3 b. 2 c. 1 d. 0

20. Let $T : R^3 \rightarrow R^3$ be the linear transformation defined by

$$T(x_1, x_2, x_3) = (x_1 + 3x_2 + 2x_3, 3x_1 + 4x_2 + x_3, 2x_1 + x_2 - x_3)$$

Then, the dimension of the null space of T is,

a. 0 b. 1 c. 2 d. 3

Answers

1. c	**2.** c	**3.** b	**4.** b
5. b	**6.** d	**7.** b	**8.** a
9. b	**10.** a	**11.** a	**12.** b
13. a	**14.** a	**15.** a	**16.** a
17. a	**18.** a	**19.** b	**20.** b

SUBJECTIVE UNSOLVED QUESTIONS
(HOTS)

1. Find the dimension of $Q(\sqrt{2}, \sqrt{3})$ over Q.

2. Show the set of functions $\{x, |x|\}$ is linearly independent in vector space of continuous functions defined in $(-1, 1)$.

3. Let $\{\alpha_1, \alpha_2, ,..., \alpha_{r-1}, \alpha_r, \alpha_{r+1}, ..., \alpha_k\}$ be a linearly independent set of k vectors in R and let $\beta_r = \sum_{j=1}^{k} a_j\alpha_j$ with $\alpha_r \neq 0$. Prove that $\{\alpha_1, \alpha_2, ..., \alpha_{r-1}, \beta_r, \beta_{r+1}, ..., \alpha_k\}$ is linearly independent.

4. If $V(F)$ be the vector space of all polynomials in x and D and T be the two linear operators on V defined by

$$D[f(x)] = \frac{df(x)}{dx}, \; T[f(x)] = xf(x)$$

for each $f(x) \in V$. Then show that the product of these operators is not commutative, *i.e.*, $DT \neq TD$ and $(TD)^2 = TD + T^2D^2$.

5. Let $f_1(x) = \sin x, f_2(x) = \cos\left(x + \frac{\pi}{6}\right)$ and $f_3(x) = \sin\left(x - \frac{\pi}{4}\right)$ for $0 \le x \le 2\pi$. Show that $\{f_1, f_2, f_3\}$ are linearly dependent by finding constants α and β such as $\alpha f_1 - 2f_2 - \beta f_3 = 0$

6. Let T be the linear map from $\mathbf{R}^3$ to $\mathbf{R}^3$ defined by $T(x, y, z) = (x + 2y - z, 2x + 3y + z, 4x + 7y - z)$.
The kernel of T is (geometrically) a ....
Whose equations(s) is (are) ....
The range of T is geometrically a ....
Whose equation(s) is (are) ....

7. The P_n be the vector space of all polynomial functions on R with degree strictly less than n. The usual basis for P_n is the set of polynomials $1, t, t^2, t^3, ..., t^{n-1}$. Define $P_3 \to P_5$ by

$$T(f(x)) = \int_0^x \int_0^u P(t)\, dt \;\; \forall\, x, u \in R.$$

a. Find matrix representation of T from P_3 to P_5 w.r.t. usual basis.
b. What is kernel of T.
c. The range space of T is spanned by which vector.

8. Let T be the reflection in the line $y = x$ in R^2.
a. Write down the standard matrix of T.
b. Use the standard matrix to compute $T(3, 4)$.

9. Consider the linear map $T : \mathbf{R}^3 \to \mathbf{R}^3$ defined by $T(x, y, z) = (x + y + z, y - 2z, y - 3z)$ and the unit sphere $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ in R^3, then find
i. $T(S)$ ii. $T^{-1}(S)$

10. Let $T : V \to V$ be a linear operator, then prove that i. $N(T) \subseteq N(T^2)$ ii. $R(T^2) \subseteq R(T)$

11. Let $T : V \to V$ be a linear operator such that rank (T^2) = rank (T), where V is finite dimensional. Prove that
i. $N(T) = N(T^2)$ ii. $R(T) = R(T^2)$ iii. $R(T^2) \cap N(T^2) = \{0\}$

Answers

1. 4 **5.** $\alpha = \sqrt{3} - 1, \beta = \sqrt{6}$ **6.** line, $\frac{-x}{5} = \frac{y}{3} = z$; plane, $2x + y - z = 0$

7. a. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1/12 \end{bmatrix}$ b. $\{0\}$ c. $\{x, x^2, x^3\}$

8. a. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ b. $(4, 3)$

9. i. $x^2 - 8xy + 26y^2 + 6xz - 38yz + 14z^2 = 1$ ii. $x^2 + 2xy + 3y^2 + 2xz - 8yz + 14z^2 = 1$

KNOW MORE

1. Which of the following is not a vector sub-space of the vector space of polynomials with real coefficient?

a. w consists of all polynomials divisible by x

b $w = \{p(x) \in V : p(3) = 0\}$

c. $w = \{p(x) \in V : p(a) = p(1-a), a \in R\}$

d. w consists of all polynomial with integral coefficient.

2. Let V be a vector space over the field F of dimension 'n'. Consider the following statements.

I. Every subset of V containing 'n' elements is a basis of V.

II. No linearly independent subset of V contains more than 'n' elements.

Which of the above statements is/are correct ?

a. (I) only b. (II) only

c. Both (I) and (II) d. neither (I) nor (II).

3. Let V be the vector space of real polynomials of degree not exceeding 2.

Let $f(x) = x - 1,$ $g(x) = x + 1$

$h(x) = x^2 - 1,$ $j(x) = x^2 + 1$

Then the set $\{f, g, h, j\}$ is

a. linearly independent b. linearly dependent $fg = h$

c. linearly dependent because $f + g - h = 0$ d. linearly dependent because $f - g - h + j = 0$

4. Consider the sub-space $w = \{[a_{ij}], a_{ij} = 0$ if i is even$\}$ of all 10×10 real matrices, then dimension of w is

a. 25 b. 50 c. 75 d. 100

5. If $T : \mathbf{R}^3 \to \mathbf{R}^3$ is given by $T(x, y, z) = (x - y, y + 3z, x + 2y)$, then T^{-1} is

a. $\frac{1}{3}\left(2x + z, -x + z, \frac{x}{3} + y - \frac{z}{3}\right)$ b. $\frac{1}{3}\left(2x + y, -x + y, \frac{1}{3}x - \frac{x}{3y} + z\right)$

c. $\frac{1}{3}\left(x + 2y, x - y, -\frac{1}{3}x + \frac{1}{3}y - z\right)$ d. $\frac{x + y + z}{3}$

6. Define T on R^2 into itself by $T(x_1, x_2) = (x_1 + x_2, x_1 - x_2)$. Then, matrix of T^{-1} relative to the standard basis for R^2 is

a. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ b $\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$ c. $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$ d. $\begin{bmatrix} -1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$

7. Let N be the vector space of all real polynomials of degree at most 3. Define, $S : N \to N$ by $(Sp)(x) = p(x+1), p \in N$. Then the matrix of S in the basis $\{1, x, x^2, x^3\}$, considered as column vectors, is given by

a. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ b $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 3 \end{bmatrix}$ d. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

8. The set of all $x \in \mathbb{R}$ for which the vectors $(1, x, 0)$, $(0, x^2, 1)$ and $(0, 1, x)$ are linearly independent in $\mathbb{R}^3$ is

a. $\{x \in \mathbb{R} : x = 0\}$ b $\{x \in \mathbb{R} : x \neq 0\}$ c. $\{x \in \mathbb{R} : x \neq 1\}$ d. $\{x \in \mathbb{R} : x \neq -1\}$

9. For any $n \in N$, let $n \in N$, P_n denotes the vector space of all polynomials with real co-efficients and of degree almost n. Define $T : P_n \to P_{n+1}$ by $T(p)(x) = p'(x) - \int_0^x p(t)\,dt$. Then, the dimension of the null space of T is

a. 0 b 1 c. n d. $n + 1$

10. Let V be the real vector space of all 2×2 real matrices. For $Q = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$ define a linear transformation T on V as $T(P) = QP$. Then the rank of T is

a. 1 b 2 c. 3 d. 4

Answers

1. d	**2.** b	**3.** d	**4.** b
5. a	**6.** c	**7.** b	**8.** c
9. a	**10.** b		

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5 Vector Spaces II

UNIT SPECIFICS

The concepts of eigen values and eigenvectors and their respective properties, eigenbases, eigenspace, symmetric, skew-symmetric matrices ,theorems based on these and orthogonal matrices, diagonalization, inner product spaces, Gram-Schmidtorthogonalization are thoroughly explained in this unit. All these topics are discussed as per students need along with sufficient number of examples.

RATIONALE

Now-a-days, matrices are typically used in statistical analysis, for many reasons. These can be used for making an algorithm in programming, as basically a database, but is more often used for statistics regarding economics and mathematics. Now, the natural frequency of the bridge is the eigen value of a system having smallest magnitude that models the bridge. Eigen value analysis is also used in the design of car stereo systems, where it helps to reproduce the vibration of the car due to the music. The application of eigen values and eigen vectors is useful for decoupling three-phase systems through symmetrical component transformation. Eigen values and eigen vectors allow us to reduce a linear operation to separate simples problems. Eigen values are not only used to explain natural occurences, but also to discover new and better design for the future. By diagonalization, we can determine the natural frequency of vibrations.

PRE-REQUISITES

1. Good knowledge of vector space, basis, dimension.
2. Deep knowledge of L.I. and L.D.
3. Student should know all the techniques to convert L.I. into matrix.
4. Aware of different types of matrices.

UNIT OUTCOMES

After completion of this unit, students will be able to:

U5-01: Relate matrices with linear transformations; compute eigen values, eigen vectors of matrices and linear transformations to diagonalize the matrices.

U5-02: Learn properties of inner product spaces and determine orthogonality w.r.t. these spaces; familiar with symmetric matrices and associated norms.

U5-03: Realize and explain the importance of adjoint of a linear transformation and its canonical form.

U5-04: Define Cauchy-Schwarz inequality and triangle inequality in inner product spaces; obtain orthonormal basis using Gram-Schmidt orthogonalisation.

MAPPING OF UNIT OUTCOMES WITH COURSE OUTCOMES

Unit 5 Outcomes	EXPECTED MAPPING WITH COURSE OUTCOMES (1- Weak Correlation; 2- Medium Correlation; 3- Strong Correlation)				
	CO-1	CO-2	CO-3	CO-4	CO-5
U5-01	–	–	1	3	3
U5-02	–	–	2	1	3
U5-03	–	–	1	3	2
U5-04	–	–	2	3	1

HISTORY

Eigen vectors gradually appeared in 18th century in solving differential equations. It may sound strange but eigen vectors (eigen functions) appeared under various names long before linear algebra, and before the word "vector" came into common use. They played central role in the theory of small oscillations. Later, Leonhard Euler studied the rotational motion of a rigid body, and discovered the importance of the principal axes. Joseph-Louis Lagrange realized that the principal axes are the eigenvectors of the inertia matrix. Cauchy also coined the term racine *caractéristique* (characteristic root), which is now called *eigen value*; his term survives in *characteristic equation*.

The term "eigenvector" comes from German, under the influence of the book of Hilbert-Courant, first edition in 1920s.

—Augustin Louis Cauchy

5.1 EIGEN VALUES AND EIGEN VECTORS OF A LINEAR OPERATOR

Let T be a linear operator on a vector space $V(F)$. If there exist a non-zero vector $v \in V$ such that $T(v) = \lambda v$ for some $\lambda \in F$, then v is called an eigen vector of T corresponding to λ and λ is called an eigen value of T corresponding to v.

The eigen vector is also called characteristic vector or latent vector and eigen value is also called characteristic value (root) or characteristic root or latent root.

5.2 EIGEN VALUES AND EIGEN VECTORS OF A MATRIX

Let A be a square matrix and 'X' is a column vector, then the matrix equation

$$AX = \lambda X$$

is equivalent to the n equations

$$\left.\begin{aligned} (a_{11}-\lambda)x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0,\\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \dots + a_{2n}x_n &= 0,\\ &\vdots\\ &\vdots\\ a_{n1}x_1 + a_{n2}x_2 + \dots + (a_{nn}-\lambda)x_n &= 0 \end{aligned}\right\} \quad \dots(1)$$

The above system of linear homogeneous equations (1) in 'n' unknowns always has the trivial solution $x_1 = x_2 = \dots = x_n = 0$.

In order to find a non-trivial solution, it is necessary that the determinant of the coefficients in (1) vanishes; *i.e.*

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \quad \dots(2)$$

It is clear that eqn. (2) is of degree n in λ's This is called the characteristic equation of the matrix A.

We can write it as

$$\lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \dots + p_{n-1}\lambda + p_n = 0 \quad \dots(3)$$

In particular, $p_1 = -(a_{11} + a_{22} + \dots + a_{nn}) = -\text{ (trace } A)$ ...(4)

and $p_n = (-1)^n \,|\, a_{ij} \,|\, = (-1)^n \text{ (det. A)}$ ...(5)

The left hand side of eqn. (3) is called the characteristic polynomial.

Equation (3) will have n-roots. These are called characteristics (or latent) roots or eigen values of the matrix A. Corresponding to each root, eqn. (1) will have a non-zero solution.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

which is known as characteristic vector or eigen vector.

If the eigen values of A are $\lambda_1, \lambda_2, \dots \lambda_n$, we can write the characteristic equation as

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \dots\dots (\lambda - \lambda_n) = 0,$$

or $\lambda^n - (\lambda_1 + \lambda_2 + \dots + \lambda_n)\lambda^{n-1} + \dots + (-1)^n \lambda_1\lambda_2 \dots \lambda_n = 0$

Comparing with eqn. (3), we can see that

$\lambda_1 + \lambda_2 + \dots + \lambda_n = -p_1 = \text{trace } A,$ by (4)

and $\lambda_1\lambda_2 \dots \lambda_n = (-1)^n p_n = \text{det. } (A),$ by (5)

Remarks:

1. Eigen values are also called as characteristic value, proper value, latent value or spectral value.
2. Similarly, Eigen vectors are also called as characteristic vector, proper vector, latent vector or spectral vector.

5.2.1 Properties of Eigen Values

a. The eigen value of a square matrix A and its transpose A' are same.

b. The sum of the eigen values of a matrix is same as the sum of the elements on the principle diagonal.

c. The product of the eigen values of a matrix A is equal to det. (A) *i.e.*, $|A|$

d. If λ is the eigen value of a non-singular matrix A, then $1/\lambda$ is an eigen value of A^{-1}.

e. If λ is an eigen value of an orthogonal matrix, then $\frac{1}{\lambda}$ is also its eigen value.

f. If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigen values of a matrix A, then A^m has the eigen values $\lambda_1^m, \lambda_2^m, ..., \lambda_n^m$ (m being a +ve integer)

g. The eigen values of idempotent matrix are either zero or unity.

h. Eigen values of the triangular matrix and diagonal matrix are same as the diagonal elements of that matrix.

Proof. d. If X be the given eigen vector corresponding to eigen value λ of A, then

$$AX = \lambda X$$

Pre-multiplying both sides by A^{-1}

$$A^{-1}(AX) = A^{-1}(\lambda X)$$

$\Rightarrow$ $$(A^{-1}A)X = \lambda A^{-1}X \quad \Rightarrow \quad IX = \lambda A^{-1}X$$

or $$\frac{1}{\lambda}X = A^{-1}X \qquad [\because \quad IX = X]$$

$\Rightarrow$ $\frac{1}{\lambda}$ is the eigen value of A^{-1}.

Proof. f. If X be the eigen vector of the eigen value λ of matrix A, then

$$AX = \lambda X$$

Pre multiplying both sides by 'A'

$$A(AX) = A(\lambda X) \quad \Rightarrow \quad (AA)X = \lambda(AX)$$

$\Rightarrow$ $$A^2X = \lambda AX$$

$\Rightarrow$ $$A^2X = \lambda^2X \qquad (\text{As } AX = \lambda X)$$

Again pre multiplying both sides by 'A', we have, in a similar way

$$A^3X = \lambda^3X$$

Continue in the same manner, we have

$$A^m \,.\, X = \lambda^m X$$

which shows that λ^m is the eigen value of A^m $(m > 0)$

Thus, we can say that if $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ be the eigen values of A, then $\lambda_1^m, \lambda_2^m, ... \lambda_n^m$ will be the eigen values of A^m.

5.2.2 Eigen Space

Let T be a linear operator on $V(F)$. If λ is an eigen value of a linear operator T, then the eigen space with respect to λ is denoted as E_λ and is defined by

$$E_\lambda = \{v : T(v) = \lambda v\}$$

For example: If
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Here $\lambda = 1, 1$ are the eigen values (as the given matrix is in triangular form) and (1, 0) is the corresponding eigen vector (In coming examples, students will learn, how to find eigen vectors for given eigen values), then $E_{\lambda=1}$ (Eigen space) = {(1, 0)}.

5.2.3 Eigen Bases

Let 'A' be a $n \times n$ matrix (or linear operator T)

Let $S = \{v_1, v_2, ..., v_n\}$ be the collection of all eigen vectors of matrix 'A'.

Then, S is said to be an eigen bases of A if S forms *a* bases of R^n.

SOME SOLVED EXAMPLES

Example 5.1. *Find the characteristic polynomial of matrix* $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ -2 & 1 & 2 \end{bmatrix}$.

Solution. Characteristic matrix of A is

$$[A - \lambda I] = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ -2 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & -1 & 1 \\ 1 & 2-\lambda & 3 \\ -2 & 1 & 2-\lambda \end{bmatrix}$$

The characteristic polynomial of matrix A is

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ 1 & 2-\lambda & 3 \\ -2 & 1 & 2-\lambda \end{vmatrix} = -\lambda^3 + 6\lambda^2 - 12\lambda + 15$$

Example 5.2. *Find all the eigen values and eigen vectors of matrix* $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Solution. The characteristic equation of the given matrix is,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(8-\lambda)((7-\lambda)(3-\lambda)-16)+6(-6(3-\lambda)+8)+2(24-2(7-\lambda))=0$

$\Rightarrow$ $(8-\lambda)(\lambda^2-10\lambda+5)+6(-10+6\lambda)+2(10+2\lambda)=0$

$\Rightarrow$ $8\lambda^2-80\lambda+40-\lambda^3+10\lambda^2-5\lambda-60+36\lambda+20+4\lambda=0$

$\Rightarrow$ $-\lambda^3+18\lambda^2-45\lambda=0$

$\Rightarrow$ $\lambda^3-18\lambda^2+45\lambda=0$ [Characteristic equation]

$\Rightarrow$ $\lambda(\lambda^2-18\lambda+45]=0$

$\Rightarrow$ $\lambda=0, \lambda^2-18\lambda+45=0$

$\Rightarrow$ $\lambda^2-15\lambda-3\lambda+45=0$

$\Rightarrow$ $\lambda(\lambda-15)-3(\lambda-15)=0$

$\Rightarrow$ $\lambda-3=0; \lambda-15=0$

$\Rightarrow$ $\lambda=3; \lambda=15; \lambda=0$

Eigen values are $\lambda = 0, 3, 15$

To find eigen vector : $[A-\lambda I]X=0$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

for $\lambda = 0$,
$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow$ $8x_1-6x_2+2x_3=0$ ...(1)

$\Rightarrow$ $-6x_1+7x_2-4x_3=0$ ...(2)

$\Rightarrow$ $2x_1-4x_2+3x_3=0$ ...(3)

Multiply eqn. (1) by 2 and adding to (2), we have

$10x_1-5x_2=0$

$\Rightarrow$ $10x_1=5x_2$

$\Rightarrow$ $x_2=2x_1$ ...(4)

Multiply eqn. (3) by 3 and adding to (2), we have

$-5x_2+5x_3=0$

$\Rightarrow$ $-5x_2=-5x_3$

$\Rightarrow$ $x_2=x_3$ ...(5)

From (4) and (5), we get

$$2x_1 = x_2 = x_3 \quad \Rightarrow \quad \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

Eigen vector corresponding to $\lambda = 0$ is $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

for $\lambda = 3$, $$\begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow$ $5x_1 - 6x_2 + 2x_3 = 0$ ...(6)

$\Rightarrow$ $-6x_1 + 4x_2 - 4x_3 = 0$ ...(7)

$\Rightarrow$ $2x_1 - 4x_2 + 0.x_3 = 0$ ...(8)

From (8), $2x_1 = 4x_2$

$\Rightarrow$ $x_1 = 2x_2$

Put $x_1 = 2x_2$ in eqn. (7), we get

$\Rightarrow$ $x_3 = -2x_2$

Thus, $x_1 = 2x_2 = -x_3$

or $$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

Eigen vectors are corresponding to $\lambda = 3$ is $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

for $\lambda = 15$

$$\begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow$ $-7x_1 - 6x_2 + 2x_3 = 0$ ...(9)

$-6x_1 - 8x_2 - 4x_3 = 0$ ...(10)

$2x_1 - 4x_2 - 12x_3 = 0$ ...(11)

Multiply eqn. (11) by 3 and adding in eqn. (10), we have

$$x_2 = -2x_3$$

Multiply eqn. (9) by 2 and adding in eqn. (10), we have

$$x_1 = -x_2$$

$\therefore$ $x_1 = -x_2 = 2x_3$

or $$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

Thus, the eigen vectors corresponding to $\lambda = 15$ is $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

Example 5.3. *Find all the eigen values, eigen vectors and of eigen basis of* $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$,

Solution. Let $$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Characteristic equation is, $|A - \lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(-2-\lambda)(-(1-\lambda)\lambda - 12) - 2(-2\lambda - 6) - 3(-4 + 1 - \lambda) = 0$

$\Rightarrow$ $(-2-\lambda)(-\lambda + \lambda^2 - 12) - 2(-2\lambda - 6) - 3(-3 - \lambda) = 0$

$\Rightarrow$ $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$\Rightarrow$ $(\lambda + 3)(\lambda + 3)(\lambda - 5) = 0$

$\Rightarrow$ $\lambda = -3, -3, 5$

Thus, eigen values are $-3, -3, 5$.

To find eigen vector

for $\lambda = -3$, eigen vector is

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} -2+3 & 2 & -3 \\ 2 & 1+3 & -6 \\ -1 & -2 & 0+3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow$ $x + 2y - 3z = 0$

$2x + 4y - 6z = 0$

$-x - 2y + 3z = 0$

From the above set, it can be seen that, $x + 2y - 3z = 0$ is only one independent equation.

So, let $z = 0$, we get $x + 2y = 0$

$\Rightarrow$ $x = -2y$

$$\frac{x}{2} = \frac{y}{-1}, z = 0$$

Hence eigen vector is $(2, -1, 0)$ for $\lambda = -3$.

Also, to find another eigen vector for $\lambda = -3$.

Let $y = 0$, we get $\quad x - 3z = 0$

$$\Rightarrow \quad x = 3z$$

$$\Rightarrow \quad \frac{x}{3} = \frac{z}{1}$$

$$\Rightarrow \quad y = 0$$

So (3, 0, 1) is another eigen vector for $\lambda = -3$.

For $\lambda = 5$, eigen vector is

$$[A - \lambda I]X = 0$$

$$\Rightarrow \quad \begin{bmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & 0-5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x + 2y - 3z = 0 \quad ...(1)$$

$$\Rightarrow \quad 2x - 4y - 6z = 0 \quad ...(2)$$

$$-x - 2y - 5z = 0 \quad ...(3)$$

Multiply eqn. (1) by 2 and adding in eqn. (2), we have

$$-12x - 12z = 0$$

or

$$-12x = 12z$$

$$\Rightarrow \quad x = -z$$

Multiply eqn. (3) by 2 and adding in eqn. (2), we have

$$y = -2z$$

Hence

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$$

So, (1, 2, –1) is eigen vector, corresponding to $\lambda = 5$.

Since all eigen vectors are L.I. and form basis of R^3.

$\therefore$ Eigen basis is {(2, –1, 0), (3, 0, 1) (1, 2, –1)}.

Example 5.4. *Find all the eigen values, eigen vectors and eigen basis of* $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

Solution. Let

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

and $\quad |A - \lambda I| = 0$ (characteristic equation)

$$\Rightarrow \quad \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (6-\lambda)\,((3-\lambda)^2 - 1) + 2(-2(3-\lambda) + 2) + 2(2 - 2(3-\lambda)) = 0$$

$$\Rightarrow \quad (6-\lambda)\,(\lambda^2 - 6\lambda + 8) + 2(2\lambda - 4) + 2(2\lambda - 4) = 0$$

$$\Rightarrow \quad -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$$

$$\Rightarrow \quad \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$
$$\Rightarrow \quad (\lambda - 2)(\lambda^2 - 10\lambda + 16) = 0$$
$$\Rightarrow \quad (\lambda - 2)(\lambda - 2)(\lambda - 8) = 0$$
$$\Rightarrow \quad \lambda = 2, 2, 8$$

Eigen values are 2, 2, 8

For $\lambda = 2$, eigen vector is

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x - 2y + 2z = 0$$
$$\Rightarrow \quad -2x + y - z = 0$$
$$2x - y + z = 0$$

Here $2x - y + z = 0$ is the only one independent equation

So, let $z = 0$

$$\Rightarrow \quad 2x - y = 0$$
or
$$2x = y$$
Thus,
$$\frac{x}{1} = \frac{y}{2}, z = 0$$

Hence (1, 2, 0) is eigen vector for $\lambda = 2$.

Again for, $\lambda = 2$ to find another eigen vector,

Let us take, $y = 0$

Here $2x - y + z = 0$ becomes $2x + z = 0$

$$\Rightarrow \quad z = -2x$$
or
$$\frac{x}{1} = \frac{z}{-2}, y = 0$$

Hence (1, 0, –2) is eigen vector.

for, $\lambda = 8$, eigen vector is,

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad -2x - 2y + 2z = 0 \quad \text{...(1)}$$
$$\Rightarrow \quad -2x - 5y - z = 0 \quad \text{...(2)}$$
$$2x - y - 5z = 0 \quad \text{...(3)}$$

Multiply eqn. (2) by 2 and adding in eqn. (1), we have

$$-6x - 12y = 0$$
$$\Rightarrow \quad x = -2y$$

Adding (2) and (3), we have

$$y = -z$$

$$\Rightarrow \quad \frac{x}{2} = \frac{y}{-1} = \frac{z}{1}$$

(2, –1, 1) is eigen vector, corresponding to $\lambda = 8$.

After that since all eigen vectors are L.I. Hence form basis of R^3.

∴ Eigen basis is {(1, 2, 0), (1, 0, –2), (2, –1, 1)}.

Example 5.5. *Find all the eigen values, eigen vectors and eigen basis of* $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

Solution. Characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)((2-\lambda)(3-\lambda)-2) - 0 - 1(2 - 4 + 2\lambda) = 0$$

$$\text{or} \quad \lambda^2 - 5\lambda + 4 - \lambda^3 + 5\lambda^2 - 4\lambda - 2\lambda + 2 = 0$$

$$\text{or} \quad \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\text{or} \quad (\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$$

$$\text{or} \quad (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

Eigen values are 1, 2, 3.

For $\lambda = 1$, eigen vector is $[A - \lambda I]X = 0$

$$\begin{bmatrix} 1-1 & 0 & -1 \\ 1 & 2-1 & 1 \\ 2 & 2 & 3-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 = 0 \quad \text{...(1)}$$

$$\Rightarrow \quad x_1 + x_2 + x_3 = 0 \quad \text{...(2)}$$

$$2x_1 + 2x_2 + 2x_3 = 0 \quad \text{...(3)}$$

Putting eqn. (1) in (2) and (3), we have

$$x_1 + x_2 = 0 \quad \Rightarrow \quad x_1 = -x_2$$

$$\text{and} \quad 2x_1 + 2x_2 = 0$$

$$\Rightarrow \quad \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$$

Thus eigen vector is $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ for $\lambda = 1$.

For $\lambda = 2$, eigen vector is, $[A - \lambda I]X = 0$

$$\begin{bmatrix} 1-2 & 0 & -1 \\ 1 & 2-2 & 1 \\ 2 & 2 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or
$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 - x_3 = 0 \quad \Rightarrow \quad x_1 = -x_3$$

$\Rightarrow$
$$x_1 + x_3 = 0$$

$$2x_1 + 2x_2 + x_3 = 0 \quad \Rightarrow \quad x_1 + 2x_2 = 0$$

$\Rightarrow$
$$\frac{x_1}{2} = \frac{x_2}{-1}\,;\ x_3 = -x_1$$

Eigen vector is $\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

For $\lambda = 3$, eigen vector is, $[A - \lambda I]X = 0$

$$\begin{bmatrix} 1-3 & 0 & -1 \\ 1 & 2-3 & 1 \\ 2 & 2 & 3-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or
$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + 0.x_2 - x_3 = 0 \quad \Rightarrow \quad -2x_1 = x_3 \qquad \text{...(1)}$$

$$2x_1 + 2x_2 + 0x_3 = 0 \quad \Rightarrow \quad x_1 = -x_2 \qquad \text{...(2)}$$

$\Rightarrow$
$$x_1 - x_2 + x_3 = 0 \qquad \text{...(3)}$$

Put $x_1 = 1$, then, $x_2 = -1$

and $x_3 = -2(1) = -2$ (from eqn. (1))

Eigen vector is $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

Since all the eigen vectors are L.I.

$\therefore$ Eigen basis is $\{(1, -1, 0), (2, -1, -2), (1, -1, -2)\}$.

Example 5.6. *Find all the eigen values, eigen vectors and eigen basis of* $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$.

Solution. Characteristic equation is, $|A - \lambda I| = 0$

i.e.,
$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (-5-\lambda)(-2-\lambda)-4=0$

or $\quad \lambda^2+7\lambda+10-4=0$

$\Rightarrow \quad \lambda^2+7\lambda+6=0$

$\Rightarrow \quad \lambda=-6,-1$

$\therefore$ Eigen values are –6 and –1

For $\lambda=-1$, eigen vectors is, $[A-\lambda I]X=0$

$$\begin{bmatrix} -5+1 & 2 \\ 2 & -2+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or
$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -4x_1+2x_2=0$

$2x_1-x_2=0$

Since $2x_1-x_2$ is the only independent equation

Now, $\quad 2x_1-x_2=0$

$\Rightarrow \quad 2x_1=x_2$

or $\quad \dfrac{x_1}{1}=\dfrac{x_2}{2}$

Eigen vector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

for $\lambda=-6$, eigen vector is given by $[A-\lambda I]X=0$

$$\begin{bmatrix} -5+6 & 2 \\ 2 & -2+6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1+2x_2=0$

$2x_1+4x_2=0$

Since x_1+2x_2 is the only independent equation.

Now, $\quad x_1+2x_2=0$

$\Rightarrow \quad x_1=-2x_2$

or $\quad \dfrac{x_1}{2}=\dfrac{x_2}{-1}$

Eigen vector is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Since all the eigen vectors are L.I.

$\therefore$ Eigen basis is $\{(1,2),(2,-1)\}$.

Example 5.7. *Find all the eigen values, eigen vectors and eigen basis of matrix* $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$.

Solution. Characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $$(3-\lambda)\,((2-\lambda)\,(5-\lambda) - 0) - 1(0) + 4(0) = 0$$

or $$(3-\lambda)\,(\lambda - 5)\,(\lambda - 2) = 0$$

$$\lambda = 2, 3, 5$$

Eigen values are 2, 3, 5 for eigen vectors.
For $\lambda = 2$, eigen vectors are $[A - \lambda I]X = 0$

$$\begin{bmatrix} 3-2 & 1 & 4 \\ 0 & 2-2 & 6 \\ 0 & 0 & 5-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

or $$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow$ $$x_1 + x_2 + 4x_3 = 0 \quad \text{...(1)}$$

$$6x_3 = 0$$

$$3x_3 = 0 \quad \Rightarrow \quad x_3 = 0 \quad \text{...(2)}$$

Putting (2) in (1), $$x_1 + x_2 = 0$$

or $$\frac{x_1}{1} = \frac{-x_2}{1}$$

$$\frac{x_1}{1} = \frac{x_2}{-1}$$

Eigen vector is $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

for $\lambda = 3$, eigen vector is, $\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$0x_1 + x_2 + 4x_3 = 0$$

$$-x_2 + 6x_3 = 0$$

$$2x_3 = 0 \quad \Rightarrow \quad x_3 = 0$$

Thus, we have $$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$$

Eigen vector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

For $\lambda = 5$, we have eigen vector is,

$$\begin{bmatrix} 3-5 & 1 & 4 \\ 0 & 2-5 & 6 \\ 0 & 0 & 5-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$2x_1 - x_2 - 4x_3 = 0$$

$$3x_2 - 6x_3 = 0 \quad \Rightarrow \quad x_2 = 2x_3$$

$$2x_1 - 2x_3 - 4x_3 = 0 \qquad \text{[putting } x_2 = 2x_3\text{]}$$

or $$2x_1 - 6x_3 = 0$$

$\Rightarrow$ $$x_1 = 3x_3$$

$\Rightarrow$ $$\frac{x_1}{3} = \frac{x_3}{1} \qquad ...(1)$$

Also, $$x_2 = 2x_3$$

$\therefore$ $$\frac{x_2}{2} = \frac{x_3}{1} \qquad ...(2)$$

From (1) and (2), $\frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}$, so the eigen vector is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Since all the eigen vectors are L.I.

$\therefore$ Eigen basis is {(1, –1, 0), (1, 0, 0), (3, 2, 1)}.

Example 5.8. *Find all the eigen values, eigen vectors and eigen basis of* $A = \begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix}$.

Solution. Characteristic equation is $|A - \lambda I| = 0$

$\Rightarrow$ $$\begin{vmatrix} 2-\lambda & 2 \\ 5 & -1-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $$(2-\lambda)(-1-\lambda) - 10 = 0$$

or $$\lambda^2 - \lambda - 12 = 0$$

$\Rightarrow$ $$(\lambda + 3)(\lambda - 4) = 0$$

$\Rightarrow$ $$\lambda + 3 = 0, \quad \lambda - 4 = 0$$

$$\lambda = -3, \quad \lambda = 4$$

Eigen values are –3 and 4.

Eigen vector for $\lambda = -3$

$$\begin{bmatrix} 2+3 & 2 \\ 5 & -1+3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$\Rightarrow \qquad 5x + 2y = 0$

and $\qquad 5x + 2y = 0$

Since there is only 1 independent equation

Thus, we have $\qquad 5x + 2y = 0$

$\Rightarrow \qquad 5x = -2y$

$\Rightarrow \qquad \dfrac{x}{-2} = \dfrac{y}{5}$

So eigen vector is $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$

For $\lambda = 4$, eigen vector is

$$\begin{bmatrix} 2-4 & 2 \\ 5 & -1-4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$-2x + 2y = 0$$

$$5x - 5y = 0$$

Again both are linearly dependent, we have

$$x - y = 0$$

$\Rightarrow \qquad x = y$

or $\qquad \dfrac{x}{1} = \dfrac{y}{1}$

So eigen vector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Since all the eigen vectors are L.I.

$\therefore \quad$ Eigen basis is {(–2, 5), (1, 1)}.

Example 5.9. *Find all the eigen values, eigen vectors and eigen bases of* $A = \begin{bmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{bmatrix}$.

Solution. (Students can try this) Eigen values are 1, –2. –2. Eigen vectors are $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. No eigen bases.

Example 5.10. *Find all the eigen values, eigen vectors and eigen bases of* $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

Solution. Eigen values are 1, 1, 5. Eigen vectors are $\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$. Eigen basis is {(–2, 1, 0), (1, 0, –1), (1, 1, 1)}.

Example 5.11. *Find all the eigen values of* $\begin{bmatrix}3 & 0\\8 & -1\end{bmatrix}$. *Hence find the eigen values of* A^{25}, *and* $A + 2I$.

Solution. Characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix}3-\lambda & 0\\8 & -1-\lambda\end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 - 2\lambda - 3 = 0$$

or $$(\lambda + 1)(\lambda - 3) = 0$$

$$\Rightarrow \quad \lambda = -1, \lambda = 3.$$

Eigen values of A^{25}

i.e., $$(-1)^{25} = -1$$

and $$(3)^{25} = 3^{25}$$

Eigen value of $A + 2I$

For $$A = 3, A + 2I = 3 + 2 = 5$$

For $$A = (-1), A + 2I = -1 + 2 = 1$$

Complex Eigen Values

Example 5.12. *Show that if* $0 < \theta < \pi$, *then* $A = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ *has no real eigen values and consequently no eigen vector.*

Solution. The characteristic equation is, $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix}\cos\theta-\lambda & -\sin\theta\\ \sin\theta & \cos\theta-\lambda\end{vmatrix} = 0$$

$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$$\cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta = 0$$

$$\lambda^2 - 2\lambda\cos\theta + 1 = 0$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \frac{2\cos\theta \pm 2i\sqrt{1-\cos^2\theta}}{2}$$

$$= \cos\theta \pm i\sin\theta$$

Hence the matrix A has no real eigen values and consequently no eigen vector.

Example 5.13. *For a given matrix* $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. *Find all the eigen values and eigen vectors of A. Is there an eigen basis for A?*

Solution. Here $$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Characteristic equation is

$$|A - \lambda I| = 0 \quad \Rightarrow \quad \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$= (1 - \lambda)^2 = 0$$

$$= \lambda = 1, 1 \text{ (Eigen values)}$$

To find eigen vector corresponding to eigen value $\lambda = 1$,

$$AX = \lambda X$$

$$[A - \lambda I]X = 0$$

$$[A - I]X = 0, (\lambda = 1)$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

[If $X = \begin{bmatrix} x \\ y \end{bmatrix}$ be the corresponding eigen vector]

$\Rightarrow$ $$y = 0$$

Take $x = 1$, then $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigen vector corresponding to $\lambda = 1$.

Here, we can see that X does not form a basis of R^2, so we do not have an eigen basis for the given matrix A.

Example 5.14. *For the linear operator* $T : R^2 \to R^2$, *find the eigen values when*

$$T(x, y) = (3x + 5y, 2x + 3y).$$

Solution. Here, $T(x, y) = (3x + 5y, 2x + 3y)$

Matrix of T relative to standard basis of R^2 is

$$A = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$$

The corresponding characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} 3-\lambda & 5 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (3 - \lambda)^2 - 10 = 0$$

$$\Rightarrow \quad \lambda = 3 \pm \sqrt{10}$$

Hence $3 + \sqrt{10}$ and $3 - \sqrt{10}$ are the eigen values of T.

Example 5.15. *For the linear operator* $T : R^2 \to R^2$, *find the eigen values, eigen vectors and eigen bases, when* $T(x, y) = (y, x)$.

Solution. Here $T(x, y) = (y, x)$

Matrix of T related to standard basis of R^2 is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The corresponding characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \qquad \lambda^2 - 1 = 0$$

or $\lambda = \pm 1$ are the eigen values of T

For $\lambda = 1$, let $X_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ be the corresponding eigen vector, then

$$AX_1 = \lambda X_1$$

$$[A - \lambda I]X_1 = 0$$

$$[A - I]\, X_1 = 0, (\lambda = 1)$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 0$$

$$\Rightarrow \qquad -x_1 + y_1 = 0$$

$$x_1 - y_1 = 0 \qquad \Rightarrow \quad x_1 = y_1$$

Taking $y_1 = 1$, we have $x_1 = 1$

$\therefore \quad X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigen vector corresponding to eigen value $\lambda = 1$

For $\lambda = -1$, let $X_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ be the corresponding eigen vector, then

$$AX_2 = \lambda X_2$$

$$\Rightarrow \qquad [A + \lambda I]X_2 = 0$$

$$[A + I]X_2 = 0 \qquad \text{(As } \lambda = -1)$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = 0$$

$$\Rightarrow \qquad x_2 + y_2 = 0$$

$$x_2 = -y_2$$

Taking $y_2 = 1, x_2 = -1$, then $X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigen vector corresponding to eigen value $\lambda = -1$.

Here, we can see that all eigen vectors form a basis of R^2.

$\therefore \quad$ Eigen bases of $T = \{(1, 1), (-1, 1)\}$.

EXERCISE 5.1

Find the characteristic roots of the following matrices:

1. $\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ **2.** $\begin{bmatrix} 2 & 3 & 11 \\ 0 & 3 & 17 \\ 0 & 0 & -2 \end{bmatrix}$ **3.** $\begin{bmatrix} a & h & g \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}$

Find the eigen values and corresponding eigen vectors for the following matrices:

4. $\begin{bmatrix} -3 & -9 & -12 \\ 1 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ **5.** $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ **6.** $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -7 & 5 & 1 \end{bmatrix}$ **8.** $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}$ **9.** $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 1 & -3 \end{bmatrix}$

10. For each of the following operator $T : R^2 \to R^2$, find the eigen values, eigen vectors and eigen bases.

i. $T(x, y) = (y, -x)$ ii. $T(x, y) = (x + 2y, 3x + 2y)$

11. For each of the following operator $T : R^3 \to R^3$, find the eigen values and bases for each eigen space.

i. $T(x, y, z) = (2x + y, y - z, 2y + 4z)$ ii. $T(x, y, z) = (x + y, y + z, -2y - z)$

iii. $T(x, y, z) = (x - y, 2x + 3y + 2z, x + y + 2z)$ iv. $T(x, y, z) = (3x + y + 4z, 2y + 6z, 5z)$

12. Let $T : R^3 \to R^3$ be a linear operator such that $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ is a matrix of T with respect to ordered basis {(1, 2, 3), (1, 2, 0), (1, 0, 0)}. Determine the eigen values, eigen vectors and eigen bases for T.

Answers

1. –1, 1, 2 **2.** 2, 3, –2 **3.** a, b, c

4. 0, (–3, 1, 0); 1, (12, –4, –1); 0, (–3, 1, 0) **5.** 1, (1, 1, –1); 2, (2, 1, 0)

6. 2; (1, 0, 0) **7.** 1; (0, 0, 1) **8.** –1(–3, –1, 3); 2(0, 1, 0); 3(1, 1, 1)

9. 2(1, 0, 0); –2(0, 1, 1); –4(0, 1, –1)

10. i. No eigen value; No eigen vector; No eigen bases

ii. Eigen values (–1, 4); Eigen vectors (–1, 1) and (2, 3); Eigen bases {(–1, 1), (2, 3)}

11. i. Eigen values (2, 3); Eigen bases {(1, 0, 0)}, {(1, 1, –2)}

ii. Eigen value (1); Eigen bases {(1, 0, 0)}

iii. Eigen values (1, 2, 3); Eigen bases {(1, 0, –1)}, {(2, –2, –1)} and {(1, –2, –1)}

iv. Eigen values (2, 3, 5); Eigen bases {(–1, 1, 0)}, {(1, 0, 0)} and {(3, 2, 1)}

12. Eigen values (2, 3, 5); Eigen vectors (–1, 1, 0), (1, 0, 0) and (3, 2, 1) respectively

Eigen bases {(–1, 1, 0), (1, 0, 0), (3, 2, 1)}

Properties Based on Transpose: Here we will write some results of transpose which are very useful in finding the solutions of Numerical Problems and in the proof of theorems.

If A' and B' denote the transpose of the matrices A and B respectively, then, following results holds:

i. $(A')' = A$

ii. $(kA)' = kA'$, k being a scalar

iii. $(A + B)' = A' + B'$, A, B being conformable for multiplication.

5.3 THEOREMS BASED ON SYMMETRIC AND SKEW-SYMMETRIC (Anti-symmetric) MATRICES

In Unit 3, we already discussed the definition of symmetric and skew-symmetric matrices.

Here we will discuss some theorems based on them:

Theorem 1: *The necessary and sufficient condition for a matrix A to be symmetric is that* $A = A'$.

Proof: The condition is necessary.

Let $A = [a_{ij}]$ be a n-rowed square symmetric matrix.

This means $a_{ij} = a_{ji}$ and A' *i.e.* the transpose of A is also n-rowed square matrix.

Now $(i, j)^{th}$ element of A'

$$= (j, i)^{th} \text{ element of } A$$

$\because$ A is symmetric $\Rightarrow$ $a_{ij} = a_{ji} \; \forall \; i, j$

$$= (i, j)^{th} \text{ element of } A$$

$\therefore$ $A = A'$

The condition is sufficient

Here $A = A'$

To prove: A is symmetric.

If $A = A'$, A must be n-rowed square matrix.

Also $(i, j)^{\text{th}}$ element of A $= (i, j)^{\text{th}}$ element of A' $[\because \; A = A']$

$$= (j, i)^{\text{th}} \text{ element of } A$$

$\therefore$ A is symmetric.

Theorem 2: *The necessary and sufficient condition for a matrix A to be skew-symmetric is that* $A' = -A$.

Proof: Let $A = [a_{ij}]$ be a n-rowed square skew-symmetric matrix. Then

$$a_{ij} = -a_{ji}.$$

Since, 'A' is n-rowed square matrix, A', $-A$ are also n-rowed square matrices.

Now $(i, j)^{\text{th}}$ element of $A' = (j, i)^{th}$ element of $(-A)$

Since 'A' is skew-symmetric $\Rightarrow$ $a_{ij} = -a_{ji} \; \forall i, j$

$$= (i, j)^{th} \text{ element of } (-A)$$

$\Rightarrow$ $A' = -A$

Conversely; If $A' = -A$, then A must be a square matrix

Also $(i, j)^{th}$ element A = the negative of the $(i, j)^{th}$ element of A' $[\because \; -A' = A]$

$$= \text{the negative of } (j, i)^{th} \text{ element of } A.$$

Hence A is a skew-symmetric matrix.

Theorem 3: *If A is a skew-symmetric and X is a column matrix, then show that $X'AX$ is a null matrix.*

Proof: Since A is a skew-symmetric matrix, then $A' = -A$

Let A be a square matrix of order n and X be a column matrix of order $n \times 1$. Now X' is a row matrix of order $1 \times n$. Hence $X'AX$ is a matrix of order 1×1.

Let $$X'AX = B \quad \text{...(1)}$$

Since B is of order 1×1, then $B' = B$, and hence B is symmetric.

Now consider $(X'AX)' = B'$

$\therefore$ $X'A'(X')' = B' \Rightarrow X'A'X'' = B'$

But $X'' = X$, $A' = -A$ and $B' = B$

$\therefore$ We have, $X'(-A)X = B$

$\Rightarrow$ $-(X'AX) = B$

$\Rightarrow$ $-B = B$ (From (1))

$\Rightarrow$ $2B = 0$

$\therefore$ $B = 0$

$\Rightarrow$ $X'AX$ is a null matrix.

Theorem 4: *Show that every square matrix can be uniquely expressed as the sum of two matrices, one symmetric and other anti-symmetric.*

Proof: Let A be a given square matrix then 'A' can be written as

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$
$$= P + Q \text{ (say)}$$

where $$P = \frac{1}{2}(A + A'),\ Q = \frac{1}{2}(A - A')$$

Now we will show that P is a symmetric matrix and Q is a skew-symmetric matrix.

For this, let $$P' = \frac{1}{2}(A + A')' = \frac{1}{2}[(A') + (A')']$$
$$= \frac{1}{2}[A' + A]$$
$$= \frac{1}{2}[A + A'] = P$$

$\therefore$ P is symmetric.

Also $$Q' = \frac{1}{2}[A - A']'$$
$$= \frac{1}{2}[A' - (A')']$$
$$= \frac{1}{2}(A' - A) = -\frac{1}{2}[A - A']$$
$$= -Q$$

$\therefore$ Q is skew-symmetric

Thus, we expressed 'A' as the sum of symmetric and skew-symmetric matrix.

To prove uniqueness:

Let $A = R + S$, where R is symmetric and S is skew-symmetric be another representation of A.

Now, $A' = (R + S)' = R' + S'$

$= R - S$ $\{\because \quad R' = R, S' = -S]$

$\therefore \quad \frac{1}{2}(A + A') = \frac{1}{2}[(R + S) + (R - S)] = R$

$\therefore \quad R = P$

and $\frac{1}{2}(A - A') = \frac{1}{2}[(R + S) - (R - S)] = S$

$\therefore \quad S = Q$

Hence the representation $A = P + Q$ is unique.

SOME SOLVED EXAMPLES

Example 5.16. *If* $A = \begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$, *then represent it as* $A = B + C$, *where* B *is symmetric and* C *is skew-symmetric.*

Solution. We have, $A = \begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$

On transposing, we get $A' = \begin{bmatrix} -1 & 2 & 5 \\ 7 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}$

On adding A and A', we get

$$A + A' = \begin{bmatrix} -2 & 9 & 6 \\ 9 & 6 & 4 \\ 6 & 4 & 10 \end{bmatrix} \qquad ...(1)$$

On subtracting A' from A, we get

$$A - A' = \begin{bmatrix} 0 & 5 & -4 \\ -5 & 0 & 4 \\ 4 & -4 & 0 \end{bmatrix} \qquad ...(2)$$

On adding (1) and (2), we have

$$2A = \begin{bmatrix} -2 & 9 & 6 \\ 9 & 6 & 4 \\ 6 & 4 & 10 \end{bmatrix} + \begin{bmatrix} 0 & 5 & -4 \\ -5 & 0 & 4 \\ 4 & -4 & 0 \end{bmatrix}$$

or $$A = \begin{bmatrix} -1 & 9/2 & 3 \\ 9/2 & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 5/2 & -2 \\ -5/2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

= Symmetric + skew-symmetric

Example 5.17. *Express* $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 7 & 1 \\ 5 & 9 & 3 \end{bmatrix}$ *as a sum of symmetric and skew-symmetric matrix.*

Solution. Let, $$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 7 & 1 \\ 5 & 9 & 3 \end{bmatrix}$$

On transposing, we have $A' = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 7 & 9 \\ 0 & 1 & 3 \end{bmatrix}$

Adding A and A', $$A + A' = \begin{bmatrix} 2 & 5 & 5 \\ 5 & 14 & 10 \\ 5 & 10 & 6 \end{bmatrix} \quad ...(1)$$

On subtracting, A' from A, we have

$$A - A' = \begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & -8 \\ 5 & 8 & 0 \end{bmatrix} \quad ...(2)$$

Adding (1) and (2), we get

$$2A = \begin{bmatrix} 2 & 5 & 5 \\ 5 & 14 & 10 \\ 5 & 10 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & -8 \\ 5 & 8 & 0 \end{bmatrix}$$

or $$A = \begin{bmatrix} 1 & 5/2 & 5/2 \\ 5/2 & 7 & 5 \\ 5/2 & 5 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -1/2 & -5/2 \\ 1/2 & 0 & -4 \\ 5/2 & 4 & 0 \end{bmatrix}$$

= Symmetric + skew-symmetric

Example 5.18. *If 'A' is a skew-symmetric matrix of odd order, or even order, then by taking an example prove that the determinant of 'A' is always '0' or real number respectively:*

Solution. (1) Let, $A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$, 'A' is odd order and skew-symmetric matrix, then

$$|A| = 0(0 + 16) - 2(0 + 12) + 3(8 + 0)$$
$$= 0 - 24 + 24 = 0$$

(2) If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *i.e.*, 'A' is even order matrix and skew-symmetric, then

$$|A| = 0 + 1 = 1 \text{ i.e., a real number.}$$

HISTORY

Inner product had its genesis in notions of orthogonal function expansions, such as the Fourier Series. Orthogonality of functions was observed first in connection with the classical Fourier Series.

So, inner product was a form on a single space. The notion of duality came decades later, only once people found that they were forced to separate the dual from the space. In 1878, Forbenius introduced the rank of matrix and used it in his definition of orthogonal matrices. Since that development, these matrices have become the backbone of the leading fields such as, cryptography, computer science, engineering, etc.

5.4 ORTHOGONAL MATRIX

A real square matrix A is called orthogonal if $AA' = A'A = I$.

5.4.1 Properties of Orthogonal Matrix

a. *If A is an Orthogonal Matrix, then* $|A| = \pm 1$.

Proof: Since determinant remains unchanged by interchanging of rows and columns, $|A'| = |A|$.

Further by definition, if A is orthogonal, then $AA' = I$

$\therefore \quad |AA'| = |I|$

$\Rightarrow \quad |A||A'| = |I|$

$\Rightarrow \quad |A|^2 = 1$

$\Rightarrow \quad |A| = \pm 1$

Remark: If A is an orthogonal square matrix of order n, since $|A| = \pm 1$, then rank of $A = n$.

b. *If A is an orthogonal matrix, then* A^{-1} *exists and is equal to* A'.

Proof: We know that if A and B are two matrices such that $AB = BA = I$, then B is called the inverse of A and the necessary and the sufficient condition for A to have inverse is $|A| \neq 0$

As seen in (a) if A is orthogonal, then $|A| = \pm 1 \neq 0$, $\quad \therefore \quad A^{-1}$ exists.

Further $\quad AA' = I, \quad \therefore \quad A^{-1}(AA') = A^{-1} . I$

$\therefore \quad (A^{-1}A) . A' = A^{-1}$

$\therefore \quad IA' = A^{-1}$

$\Rightarrow \quad A' = A^{-1}$

c. *If A and B are two orthogonal square matrices of order n, then AB and BA are also orthogonal.*

Proof: Since A and B are square matrices of order n. AB and BA are defined and are square matrices of order n.

Since A, B are orthogonal,

$$|A| \neq 0, |B| \neq 0 \ (= \pm 1) \text{ and } A^{-1}, B^{-1} \text{ exists}$$

Further, $\quad |AB| = |A||B| \neq 0$

$\therefore$ $(AB)^{-1}$ exists.

Now $$(AB)' = B'A'$$

Further $$(AB)'(AB) = B'A'AB = B'(A'A)B = B'IB = B'B = I$$

Hence $(AB)'$ is the inverse of AB

$\therefore$ AB is orthogonal.

Note: If A is an orthogonal matrix, then $|A| = \pm 1$

If $|A| = 1$, then A is called proper orthogonal matrix.

Remark: Numerical analysis takes advantage of many of the properties of **orthogonal matrices** for numerical linear algebra. For **example**, it is often desirable to compute an **orthonormal** basis for an inner product space, or an **orthogonal** change of bases; both take the form of **orthogonal matrices**.

SOME SOLVED EXAMPLES

Example 5.19. *Verify that* $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ *is orthogonal.*

Solution. Here $$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

$\therefore$ $$A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

and $$AA' = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence 'A' is an orthogonal matrix.

Example 5.20. *Determine the values of* α, β, γ *when* $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ *is orthogonal.*

Solution. Let $$A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$$

On transposing A, we have

$$A' = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$$

If A is orthogonal, then $AA' = I$

$$\therefore \qquad AA' = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}\begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+4\beta^2+\gamma^2 & 0+2\beta^2-\gamma^2 & 0-2\beta^2+\gamma^2 \\ 0+2\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 \\ 0-2\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \qquad 4\beta^2 + \gamma^2 = 1 \qquad$ ...(1)

and $\qquad -2\beta^2 + \gamma^2 = 0 \quad \Rightarrow \quad \beta^2 = \dfrac{\gamma^2}{2} \qquad$ ...(2)

(1) $\Rightarrow \qquad 4 \,.\, \dfrac{\gamma^2}{2} + \gamma^2 = 1 \qquad \left(\text{Putting } \beta^2 = \dfrac{\gamma^2}{2}\right)$

or $\qquad 2\gamma^2 + \gamma^2 = 1 \quad \Rightarrow \quad 3\gamma^2 = 1 \quad \Rightarrow \quad \gamma = \pm \dfrac{1}{\sqrt{3}}$

Also $\qquad \beta^2 = \dfrac{1}{6} \quad$ (from (2)) $\quad \Rightarrow \quad \beta = \pm \dfrac{1}{\sqrt{6}}$

Also, we have $\qquad \alpha^2 = 1 - \beta^2 - \gamma^2 \qquad [\because \quad \alpha^2 + \beta^2 + \gamma^2 = 1]$

$$= 1 - \frac{1}{6} - \frac{1}{3}$$

$$\alpha^2 = \frac{6-1-2}{6} = \frac{1}{2}$$

$\Rightarrow \qquad \alpha = \pm \dfrac{1}{\sqrt{2}}$

Example 5.21. *If* $A = \dfrac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, *prove that* $A^{-1} = A'$, *A' being the transpose of A.*

Solution. We have, $A = \dfrac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, and $A' = \dfrac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$, then

$$AA' = \frac{1}{81}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 64+1+16 & -32+4+28 & -8-8+16 \\ -32+4+28 & 16+16+49 & 4-32+28 \\ -8-8+16 & 4-32+28 & 1+64+16 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \qquad AA' = I \quad \Rightarrow \quad A' = A^{-1}$$

Example 5.22. *Is the matrix* $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$ *orthogonal?*

Solution. Transpose of A is $A' = \begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix}$

Consider
$$A'A = \begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix} \times \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 4+16+9 & -6+12-3 & 2+4-27 \\ -6+12-3 & 9+9+1 & -3+3+9 \\ 2+4-27 & -3+3+9 & 1+1+81 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & 3 & -21 \\ 3 & 19 & 9 \\ -21 & 9 & 83 \end{bmatrix} \neq I$$

So, matrix A is not orthogonal.

Example 5.23. *Is the matrix* $A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ *orthogonal?*

Solution. The transpose matrix $A' = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

So,
$$A'A = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \times \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta+0+\sin^2\theta & 0+0+0 & \cos\theta\sin\theta-\cos\theta\sin\theta \\ 0+0+0 & 0+1+0 & 0+0+0 \\ \cos\theta\sin\theta-\cos\theta\sin\theta & 0+0+0 & \sin^2\theta+0+\cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence the given matrix is orthogonal.

EXERCISE 5.2

1. If $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 5 & 0 \end{bmatrix}$, then find $(AB)'$. Hence verify also $(AB)' = B'A'$.

2. If $A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$, then show that AA' and $A'A$ are both symmetric matrices.

3. If A and B are symmetric matrices, then show that $AB - BA$ is a skew-symmetric matrix.

4. Express the matrix 'A' given below as the sum of a symmetric and a skew-symmetric matrix.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix}$$

5. Prove that the following matrix is orthogonal and hence find A^{-1}.

$$A = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$$

6. If $A = \begin{bmatrix} 1/3 & 2/3 & a \\ 2/3 & 1/3 & b \\ 2/3 & -2/3 & c \end{bmatrix}$ is orthogonal, find a, b and c.

7. Check the given matrix is orthogonal or not?

$$A = \begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$$

8. Is the following matrix orthogonal?

$$A = \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$$

9. Prove that the following matrices are orthogonal and hence find A^{-1} also.

i. $\frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & -\sqrt{3} \end{bmatrix}$ ii. $\frac{1}{\sqrt{6}}\begin{bmatrix} 0 & \sqrt{3} & \sqrt{3} \\ 2 & 1 & -1 \\ \sqrt{2} & -\sqrt{2} & \sqrt{2} \end{bmatrix}$ iii. $\frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 & 0 \\ 1 & \sqrt{3} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

10. If $3A = \begin{bmatrix} a & b & c \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ and if A is orthogonal, find a, b, c.

11. Express the matrix $A = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

12. Check whether the following matrices are proper orthogonal or improper orthogonal:

i. $\begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix}$ ii. $\begin{bmatrix} 12/13 & 5/13 \\ -5/13 & 12/13 \end{bmatrix}$ iii. $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$

Answers

1. $\begin{bmatrix} 17 & 4 \\ 0 & -2 \end{bmatrix}$

4. Symmetric matrix = $\begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 5 & 9/2 \\ 3/2 & 9/2 & 3 \end{bmatrix}$; Skew-symmetric matrix = $\begin{bmatrix} 0 & 2 & 5/2 \\ -2 & 0 & -3/2 \\ -5/2 & 3/2 & 0 \end{bmatrix}$

5. A' is the inverse of A

6. $a = \pm\frac{2}{3}, b = \pm\frac{2}{3}, c = \pm\frac{1}{3}$

7. No

8. No

9. i. $\frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & -2 & 1 \\ \sqrt{3} & 0 & -\sqrt{3} \end{bmatrix}$ ii. $\frac{1}{\sqrt{6}}\begin{bmatrix} 0 & 2 & \sqrt{2} \\ \sqrt{3} & 1 & -\sqrt{2} \\ \sqrt{3} & -1 & \sqrt{2} \end{bmatrix}$ iii. $\frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 & 0 \\ -1 & \sqrt{3} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

10. $a = 2, b = 2, c = 1$

11. Symmetric Matrix = $\begin{bmatrix} 3 & 0 & 11/2 \\ 0 & 7 & 3/2 \\ 11/2 & 3/2 & 0 \end{bmatrix}$; skew-symmetric Matrix = $\begin{bmatrix} 0 & -2 & 1/2 \\ 2 & 0 & -5/2 \\ -1/2 & 5/2 & 0 \end{bmatrix}$

12. i. Improper ii. Proper iii. Not Orthogonal

5.5 DIAGONALIZATION OF LINEAR OPERATOR

Consider a linear operator $T : V \to V$. Then T is said to be diagonalizable if it can be represented by a diagonal matrix D. Thus, T is diagonalizable if and only if there exists a basis B of V such that matrix of T with respect to B is a diagonal matrix D.

5.5.1 Diagonalization of Matrices

Let $x_1, x_2, ..., x_n$ be the eigen vectors of a square matrix A, corresponding to the eigen values $\lambda_1, \lambda_2, ..., \lambda_n$, respectively, then

$$Ax_i = \lambda_i x_i,$$

Denote by 'B' the square matrix whose columns are $x_1, x_2, ..., x_n$. For bravity, we shall write P as $[x_1, x_2, ..., x_n]$, then

$$\begin{aligned} AP &= A[x_1\ x_2\ ...\ x_n] \\ &= [Ax_1\ Ax_2\ ...\ Ax_n] \\ &= [\lambda_1 x_1\ \lambda_2 x_2\ ...\ \lambda_n x_n] \\ &= PD \end{aligned}$$

[This step can be verified by writing P and D as full and multiplying]

where
$$D = \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n \end{bmatrix}$$

This gives $\quad P^{-1} AP = D$

This method fails when 'n' linearly independent eigen vectors do not exist for A. Such matrices are not diagonalizable.

When n linearly independent eigen vectors exist, P is non-singular and $P^{-1} AP$ is a diagonal matrix with eigen values as its diagonal elements.

The matrix P which is used to diagonalize the matrix A is called the modal matrix of A and the diagonal matrix thus obtained is known as spectral matrix of A.

Two matrices A and C are said to be similar if there is a non-singular matrix B such that

$$C = B^{-1} AB$$

Evidently, a diagonalizable matrix A is similar to a diagonal matrix D.

Note: Similar matrices have the same eigen values.

INTERESTING FACTS

1. It simplifies significantly certain computations. However, the first use of diagonalization can be seen in **Markov processes**, where the power of some square matrix are used extensively and Markov processes are really rich in applications, such as
   - Market
   - Weather forecasting
   - Genetics
   - Diffusion of gasses
   - The most famous one is probably Google's page ranking algorithm.
2. It is also use in Mechanics, for example, a way to find principal axes of inertia (with tensor of inertia being the diagonalized matrix).
3. One other thing is finding normal modes of an oscillating system (which requires simultaneous diagonalization of two matrices of kinetic and potential energy

VIDEO REFERENCES

Eigenvalues & Eigenvectors

Method to Find Eigenvalues and Eigenvectors, Diagonalization of Matrices

APPLICATIONS TO REAL LIFE

- One important use of diagonalisation is for computing higher powers of matrix efficiently.
  If $A = M^{-1} DM$, then $A^n = M^{-1} D^n M$
  The above property makes it easy to compute higher powers of matrix A, since computing D^n is much more easy as compared with computing A^n.

SOME SOLVED EXAMPLES

Example 5.24. *Diagonalize* $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ *and obtain the modal matrix.*

Solution. The characteristic equation of the given matrix is $|A - \lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (-1-\lambda)\,[-\lambda(2-\lambda)+1] - 2[-\lambda+1] - 2[-1+2-\lambda] = 0$$
$$\Rightarrow \quad (-1-\lambda)\,[\lambda^2 - 2\lambda + 1] - 2[-\lambda+1] - 2\,[-\lambda+1] = 0$$
$$\Rightarrow \quad (-1-\lambda)\,(1-\lambda)^2 - 4(1-\lambda) = 0$$
$$\Rightarrow \quad (1-\lambda)\,[(-1-\lambda)\,(1-\lambda) - 4] = 0$$
$$\Rightarrow \quad (1-\lambda)\,(\lambda^2 - 5) = 0$$
$$\Rightarrow \quad \lambda = 1, \pm\sqrt{5}$$

Now, we find the eigenvectors corresponding to these eigen values.

For $\lambda = 1$, let $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ be an eigen vector such that

$$AX_1 = \lambda X_1$$
$$\Rightarrow \quad (A - \lambda I)X_1 = 0$$
$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + 2y_1 - 2z_1 = 0 \quad \Rightarrow \quad -x_1 + y_1 - z_1 = 0 \qquad \text{...(1)}$$

$$x_1 + y_1 + z_1 = 0 \qquad \text{...(2)}$$

$$-x_1 - y_1 - z_1 = 0 \quad \Rightarrow \quad x_1 + y_1 + z_1 = 0$$

(1) + (2) $\Rightarrow$ $2y_1 = 0 \quad \Rightarrow \quad y_1 = 0$

From (2) $-x_1 - z_1 = 0 \quad \Rightarrow \quad -x_1 = z_1$

Take $z_1 = -1, \quad \Rightarrow \quad x_1 = 1$

$\therefore$ $X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

For $\lambda = \sqrt{5}$, let $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ be an eigenvector such that

$$[A - \sqrt{5}I]\, X_2 = 0$$

$$\begin{bmatrix} -1-\sqrt{5} & 2 & -2 \\ 1 & 2-\sqrt{5} & 1 \\ -1 & -1 & 0-\sqrt{5} \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(-1-\sqrt{5})\, x_2 + 2y_2 - 2z_2 = 0 \qquad \text{...(3)}$$

$$x_2 + (2-\sqrt{5})\, y_2 + z_2 = 0 \qquad \text{...(4)}$$

$$-x_2 - y_2 - \sqrt{5}\, z_2 = 0 \qquad \text{...(5)}$$

(4) + (5) $\Rightarrow$ $(1-\sqrt{5})y_2 + (1-\sqrt{5})z_2 = 0$ ...(6)

$\Rightarrow$ $y_2 = -z_2$

Take $z_1 = 1$, then $y_2 = -1$

From (5) $-x_2 = y_2 + \sqrt{5}z_2 = -1 + \sqrt{5}$

$\Rightarrow$ $x_2 = 1 - \sqrt{5}$

$\therefore$ $X_2 = \begin{bmatrix} 1-\sqrt{5} \\ -1 \\ 1 \end{bmatrix}$

For $\lambda = -\sqrt{5}$, let $X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$ be an eigenvector such that

$$\begin{bmatrix} -1+\sqrt{5} & 2 & -2 \\ 1 & 2+\sqrt{5} & 1 \\ -1 & -1 & 0+\sqrt{5} \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(-1+\sqrt{5})\, x_3 + 2y_3 - 2z_3 = 0 \qquad \text{...(7)}$$

$$x_3 + (2+\sqrt{5})y_3 + z_3 = 0 \quad \text{...(8)}$$

$$-x_3 - y_3 + \sqrt{5}\, z_3 = 0 \quad \text{...(9)}$$

(8) + (9) $\Rightarrow$ $(1+\sqrt{5})y_3 + (1+\sqrt{5})z_3 = 0$

$\Rightarrow$ $y_3 = -z_3$

Take $z_3 = 1, y_3 = -1$

From (9) $-x_3 = y_3 - \sqrt{5}\ z_3$

$\Rightarrow$ $-x_3 = -1 - \sqrt{5}$

$\therefore$ $X_3 = \begin{bmatrix} 1+\sqrt{5} \\ -1 \\ 1 \end{bmatrix}$

$\therefore$ Modal matrix $P = \begin{bmatrix} 1 & 1-\sqrt{5} & 1+\sqrt{5} \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

$$|P| = -2\sqrt{5}$$

$$P^{-1} = \frac{1}{(-2\sqrt{5})}\begin{bmatrix} 0 & 1 & -1 \\ 2\sqrt{5} & 2+\sqrt{5} & -2+\sqrt{5} \\ 2\sqrt{5} & 1 & -1 \end{bmatrix}^T = \frac{1}{(-2\sqrt{5})}\begin{bmatrix} 0 & 2\sqrt{5} & 2\sqrt{5} \\ 1 & 2+\sqrt{5} & 1 \\ -1 & -2+\sqrt{5} & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & -1 \\ \dfrac{-1}{2\sqrt{5}} & \dfrac{-1}{\sqrt{5}} - \dfrac{1}{2} & \dfrac{-1}{2\sqrt{5}} \\ \dfrac{1}{2\sqrt{5}} & \dfrac{1}{\sqrt{5}} - \dfrac{1}{2} & \dfrac{1}{2\sqrt{5}} \end{bmatrix}$$

$$P^{-1}AP = D$$

$\Rightarrow$
$$D = \begin{bmatrix} 0 & -1 & -1 \\ \dfrac{-1}{2\sqrt{5}} & \dfrac{-1}{\sqrt{5}} - \dfrac{1}{2} & \dfrac{-1}{2\sqrt{5}} \\ \dfrac{1}{2\sqrt{5}} & \dfrac{1}{\sqrt{5}} - \dfrac{1}{2} & \dfrac{1}{2\sqrt{5}} \end{bmatrix}\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1-\sqrt{5} & 1+\sqrt{5} \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & -1 \\ \dfrac{-1}{2} & \dfrac{-5-2\sqrt{5}}{2\sqrt{5}} & \dfrac{-1}{2} \\ \dfrac{-1}{2} & \dfrac{5-2\sqrt{5}}{2\sqrt{5}} & \dfrac{-1}{2} \end{bmatrix}\begin{bmatrix} 1 & 1-\sqrt{5} & 1+\sqrt{5} \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}.$$

Example 5.25. *Show that the matrix* $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ *is not diagonalizable over the field C.*

Solution. Given $$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Corresponding characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda = 2, 2$$

Thus the only distinct eigen value is 2

If $X = \begin{bmatrix} x \\ y \end{bmatrix}$ be the corresponding eigen vector, then

$$AX = \lambda X$$

or $$[A - \lambda I]X = 0$$

For $\lambda = 2$, $$[A - 2I]X = 0$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 0x + y = 0 \quad \Rightarrow \quad y = 0$$

Taking $x = 1, y = 0, X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the only eigen vector corresponding to $\lambda = 2$.

Thus the given square matrix A has only one linearly independent eigen vector.

So the given square matrix A is not diagonalizable.

[For the given matrix A to be diagonalizable, it must have 2 linearly independent eigen vectors]

Example 5.26. *Show that the matrix* $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ *is diagonalizable.*

Solution. First we will find the eigen values and eigen vectors in the same manner as we have done in Example 5.4 on page no. 332.

After that, to check for diagonalizability.

Since, the given matrix A has 3 linearly independent eigen vectors.

So the given matrix is diagonalizable.

Example 5.27. *Check the diagonalizability of the given matrix* $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

Solution. Proceeding in a similar way as in Example 5.5 on page no. 334.

Then, to check for diagonalization

Since the given matrix A has 3 linearly independent eigen vectors.

So the given matrix is diagonalizable,

Example 5.28. *For the linear operator $T : R^2 \to R^2$ given by $T(x, y) = (-5x + 2y, 2x - 2y)$. Check the diagonalizability of T?*

Solution. Given $T(x, y) = (-5x + 2y, 2x - 2y)$

Matrix associated with standard basis of R^2 is $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

Now proceed in a similar way as in Example 5.6 on page no. 335.

After that, since T has 2 linearly independent eigen vectors.

So T is diagonalizable.

[In a similar way, students can practice many more questions based on the concept of diagonalizability]

EXERCISE 5.3

1. Show that the given matrices are diagonalizable:

i. $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ ii. $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

2. Show that the given matrix $A = \begin{bmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{bmatrix}$ is not diagonalizable.

3. Check the diagonalizability of $A = \begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix}$.

4. Whether the given matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ is diagonalizable or not? Support your answer by giving proper reason.

Answers

3. Diagonalizable

4. Yes, Diagonalisable, as A.M. of each eigen value = G.M. of each eigen value.

5.6 INNER PRODUCT SPACE

A vector space together with an inner product defined on it is called an inner product space.

The inner product of two vectors $\vec{u}$ and $\vec{v}$ is denoted as $<u, v>$ or (u, v)

We know that scalar product of two vector $\vec{u}$ and $\vec{v}$ in R^n. In a similar way, we can define an inner product of two-column vectors $\vec{u}$ and $\vec{v}$ as

$$<u, v> = u^T . v$$

This definition can be extended to general real vector spaces by taking basic property of $\vec{u}$, $\vec{v}$

Let u and v be two column vectors over the real field R

$$u = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ and } v = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

then $\quad <u, v> = u^T v$

$$= (x_1, x_2, ..., x_n) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$= (x_1 y_1 + x_2 y_2 + ... + x_n y_n)$$

It is called the standard inner product for R^n.

Definition: Let V be a vector space over the field F. Let $a, b \in F$ be arbitrary scalars and $u, v, w \in V$ are arbitrary vectors. The vector space V is called an inner product space if there exist a function.

$<, > V \times V \to F$ satisfying the following axioms

1. $<u, v> = \overline{<v, u>}$, *i.e.*, complex conjugate of (v, u)
2. $<u, u> \geq 0$ and $<u, u> = 0$ iff $u = 0$.
3. $<au + bv, w> = a < u, w > + b < v, w >$

The function $<, >$ satisfying properties (1), (2), (3) is called an inner product on V.

Remarks: 1. A real inner product space is called Euclidean space and a complex inner product space is called unitary space.

2. If $F = R$ (Field becomes Real), then axiom. (1) simply states that

$$<u, v> = <v, u> \text{ (symmetric property)}$$

3. Since $<u, v> = \overline{<v, u>}$

$\therefore \quad <u, u> = \overline{<u, u>}$

which shows that $<u, u>$ is a real number and so axiom (2) makes sense.

5.6.1 Properties of Inner Product Space

Let V be an inner product space. Show that if $a, b, c \in F$ are arbitrary scalars and $u, v, w \in V$ are arbitrary vectors, then following are true:

i. $<au, v> = a <u, v>$

ii. $<u, av> = \bar{a} <u, v>$

iii. $<u, bv + cw> = \bar{b} < u, v > + \bar{c} <u, w>$

iv. $<o, v> = 0$

v. $<u, v> = 0 \ \forall \ v \in V \Rightarrow u = 0$

Solution: i. We know that for an inner product space,

$$<au + bv, w> = a <u, w> + b <v, w>$$

Putting $b = 0$ in (1) ...(1) [by axiom (3)]

We get,

$$<au, w> = a <u, w> + 0 <v, w>$$
$$= a <u, w>$$

$\therefore$ $$<au, v> = a <u, v>$$

ii. $$<v, au> = \overline{<au, v>}$$
$$= [\overline{a <u, v>}]$$
$$= \bar{a} \overline{<u, v>}$$
$$= \bar{a} <v, u> \quad \text{(By axiom (1))}$$

Interchanging u and v, we get

$$<u, av> = \bar{a} <u, v>$$

iii. $$<u, bv + cw> = \overline{<bv + cw, u>}$$
$$= [\overline{b <v, u> + c <w, u>}]$$
$$= \bar{b} \overline{<v, u>} + \bar{c} \overline{<w, u>}$$
$$= \bar{b} <u, v> + \bar{c} <u, w> \quad \text{(By axiom (1))}$$

iv. $$<0, v> = 0$$

(Students can try it)

v. $$<u, v> = 0 \ \forall \ v \in V$$

or $$<u, u> = 0 \quad \text{(By taking } v = u)$$

$\Rightarrow$ $$u = 0 \quad \text{(using axiom (2))}$$

5.6.2 Length (Norm) of a Vector

Let V be an inner product space. For any vector $v \in V$, the norm of v is defined as $\sqrt{<v, v>}$ and is denoted as $|| v ||$.

i.e., $$|| v || = \sqrt{<v, v>}$$

Remark: 1. *A* vector of norm 1 is called a unit vector
2. The distance between the vectors u and v is denoted by $d(u, v)$ and is defined as

$$d(u, v) = || u - v ||$$

SOME SOLVED EXAMPLES

Example 5.29. *Let* $u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ *and* $v = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ *be two column vectors. Find their inner product and length of each vector.*

Solution. Given $u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $v = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$,

then $$<u, v> = u^T v$$
$$= [1 \ \ 2 \ \ 3] \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$
$$= (1)(2) + (2)(-1) + (3)(1)$$
$$= 3 \text{ (inner product of } u \text{ and } v)$$

and $$\| u \| = \sqrt{<u, u>}$$

$\Rightarrow$ $$\| u \|^2 = <u, u>$$
$$= u^T . u$$
$$= [1 \ \ 2 \ \ 3] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$
$$= (1)(1) + (2)(2) + (3)(3)$$
$$= 1 + 4 + 9 = 14$$

then $$\| u \| = \sqrt{14}$$

and $$\| v \| = \sqrt{<v, v>}$$

or $$\| v \|^2 = <v, v>$$
$$= v^T v$$
$$= [2 \ \ -1 \ \ 1] \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$
$$= (2)(2) + (-1)(-1) + (1)(1) = 4 + 1 + 1 = 6$$

$\Rightarrow$ $$\| v \| = \sqrt{6}$$

Example 5.30. *Suppose V(R) is a vector space of all polynomials over the unit interval* $0 \le t \le 1$. *If* $f(t)$, $g(t) \in V$ *and inner product on V is defined by* $<f(t), g(t)> = \int_0^1 f(t)\, g(t)\, dt$, *then find* $<f, g>$ *and* $\| g \|$, *if* $f(t) = t^2 + t - 4$ *and* $g(t) = t - 1$.

Solution. Given

$$<f(t), g(t)> = \int_0^1 f(t)\, g(t)\, dt$$

then, $$= \int_0^1 (t^2 + t - 4)(t - 1)\, dt$$
$$= \int_0^1 (t^3 - 5t + 4)\, dt$$

$$= \left| \frac{t^4}{4} - \frac{5t^2}{2} + 4t \right|_0^1$$

$$= \frac{1}{4} - \frac{5}{2} + 4$$

$$= \frac{7}{4}$$

Also, $\quad || g ||^2 = <g, g>$

$$= \int_0^1 g(t) \, . \, g(t) \, dt$$

$$= \int_0^1 (t-1)^2 \, dt$$

$$= \left| \frac{(t-1)^3}{3} \right|_0^1$$

$$= \frac{1}{3}$$

Thus, $\quad || g || = \dfrac{1}{\sqrt{3}}$

Example 5.31. *Let V be a vector space over C of all complex valued continuous functions on the interval [α, β]. Define the inner product on V as <f(t), g(t)> =* $\int_\alpha^\beta f(t) \, . \, \overline{g(t)} \, dt$ *. Prove that V is an inner product space.*

Solution. i. Given,

$$<f(t), g(t)> = \int_\alpha^\beta f(t) \, . \, \overline{g(t)} \, dt$$

We can write it as,

$$< \overline{g(t) \, . \, f(t)} > = \left[\overline{\int_\alpha^\beta g(t) \, \overline{f(t)} \, dt} \right]$$

$$= \int_\alpha^\beta \overline{g(t)} \, . \, f(t) \, dt$$

$$= \int_\alpha^\beta f(t) \, . \, \overline{g(t)} \, dt$$

Thus, $\quad <f(t) \, . \, g(t)> = < \overline{g(t) \, . \, f(t)} >$

ii. Also $\quad <f(t), f(t)> = \int_\alpha^\beta f(t) \, . \, \overline{f(t)} \, dt$

$$= \int_\alpha^\beta [f(t)]^2 \, dt$$

and $\quad <f(t), f(t)> = 0$ iff $| f(t)|^2 = 0$

iff $| f(t) | = 0$

iii. Now, $< a f(t) + bg(t), h(t)>$

$$= \int_{\alpha}^{\beta} [af(t)+bg(t)]\overline{h(t)}\,dt$$

$$= a\int_{\alpha}^{\beta} f(t)\overline{h(t)}\,dt + b\int_{\alpha}^{\beta} g(t)\overline{h(t)}\,dt$$

$$= a <f(t), h(t)> + b <g(t), h(t)>$$

Thus, V is an inner product space

> **Remark:** If V is the vector space of all continuous real valued functions, then the above definition takes the form $<f(t), g(t)> = \int_{\alpha}^{\beta} f(t)\,g(t)\,dt$.

Some important theorems whose results are used in many places and students must know these when they are studying the topic Inner Product space.

1. Cauchy's Schwarz's Inequality

Statement: Let V be an inner product space. Then $| <u, v> | \leq || u || \, || v || \ \forall \ u, v \in V$

Proof. If $u = 0$, then $<u, v> = <0,v> = 0$

and $|| u || = \sqrt{<u, u>} = \sqrt{<0, 0>} = 0$

Similarly, the inequality is valid, when $v = 0$.

Let $u \neq 0$, then $|| u || \neq 0$, as

$$|| u || = 0 \Rightarrow \sqrt{<0, 0>} = 0$$

$$\Rightarrow \quad <u, u> = 0 \Rightarrow u = 0$$

Let
$$w = v - \frac{<v, u> u}{||u||^2} \qquad ...(1)$$

Then
$$<w, u> = <\left(v - \frac{<v, u> u}{||u||^2}, u\right)>$$

$$= <v, u> - <\left(\frac{<v, u> u}{||u||^2}\right), u>$$

$$= <v, u> - \frac{<v, u>}{||u||^2} <u, u> \qquad [\because \ <au, v> = a <u, v>]$$

$$= <v, u> - \frac{<v, u>}{||u||^2} || u ||^2$$

$$= <v, u> - <v, u>$$

$$<w, u> = 0 \qquad ...(2)$$

Also
$$|| w ||^2 = <w, w> = <\left(v - \frac{<v, u> u}{||u||^2}, w\right)>$$

$$= <v, w> - \frac{<v, u><u, w>}{||u||^2}$$

$$= < v, w > - \frac{< v, u >}{\|u\|^2} \overline{< w, u >}$$

$$= < v, w > - \frac{< v, u >}{\|u\|^2} . \bar{0} \qquad \text{[By (2)]}$$

$$= <v, w> - 0 = <v, w>$$

$$= < v, v - \frac{< v, u > u}{\|u\|^2} > \qquad \text{[By (1)]}$$

$$= < v, v > - \frac{\overline{< v, u >}}{\|u\|^2} . < v, u >$$

$$= \|v\|^2 - \frac{|< v, u >|^2}{\|u\|^2} \qquad [\because \ \bar{z} . z = |z|^2]$$

Now $\qquad \| w \|^2 \geq 0$

$\Rightarrow \quad \dfrac{\|v\|^2 \|u\|^2 - |< v, u >|^2}{\|u\|^2} \geq 0$

$\Rightarrow \quad \| u \|^2 \| v \|^2 - | <u, v>|^2 \geq 0$

$\Rightarrow \quad \| u \| \| v \| \geq | <u, v> |$ [By taking square root an both sides]

or $\quad | <u, v> | \leq \| u \| \| v \|$

> **Remark:** Let $u = (a_1, a_2)$ and $v = (b_1, b_2)$ be arbitrary members of C^2.
> Define $<u, v> = u . v = a_1\overline{b_1} + a_2\overline{b_2}$
> It is an inner product space on C^2.

2. Triangle Inequality

Statement: Let V be an inner product space. Then $\| u + v \| \leq \| u \| + \| v \|$.

Proof: We know,

$$\| u + v \|^2 = <u + v, u + v>$$

$$= <u, u + v > + <v, u + v>$$

$$= <u, u> + <u, v> + <v, u> + <v, v>$$

$$= \| u \|^2 + <u, v> + \overline{<u, v>} + \| v \|^2$$

$= \| u \|^2 + \| v \|^2 + 2Re <u, v>$ where $Re <u, v>$ = real part of $<u, v>$

$\leq \| u \|^2 + \| v \|^2 + 2 \| u \| \| v \| \qquad [\because \ Re <u, v> \leq | <u, v> |$ and $| <u, v> | \leq \| u \| \| v \|]$

$$\leq [\| u \| + \| v \|]^2$$

or $\quad \| u + v \|^2 \leq [\| u \| + \| v \|]^2$

Thus $\quad \| u + v \| \leq \| u \| + \| v \|$

5.6.3 Orthogonal Vectors (Perpendicular Vector)

Let V be an inner product space. A vector $u \in V$ is said to be orthogonal to $v \in V$ if

$$<u, v> = 0$$

5.6.4 Orthonormal Vectors

Two vectors u and v of an inner product space V are said to be orthonormal if

i. $<u, v> = 0$

ii. The norm of each vector u and v is 1.

> **Remarks:** i. $0 \perp u \ \forall \ u \in V$
> ii. $u \perp u$, iff $u = 0$, where $u \in V$
> iii. $u \perp v \Rightarrow v \perp u$ for $u, v \in V$
> iv. $u \perp v \Rightarrow \alpha u \perp v$, for any scalar $\alpha \in F$ and $u, v \in V$.

SOME SOLVED EXAMPLES

Example 5.32. *If the vectors $u_1 = (1, 2i, i)$, $u_2 = (0, 1 + i, 1)$, $u_3 = (2, 1 - i, i) \in C^3$, then*

i. *find the norm (length) of each vector u_i.*

ii. *show that the vector $v = (1 - i, -1, 1 - i)$ is orthogonal to both u_1 and u_2.*

iii. *obtain a vector which is orthogonal to both u_1 and u_3.*

Solution. i. The norm of vector u_i is given by

$$\| u_1 \| = \sqrt{<u_1, u_1>} = \sqrt{(1)(1) + (2i)(-2i) + (i)(-i)}$$
$$= \sqrt{6}$$
$$\| u_2 \| = \sqrt{<u_2, u_2>} = \sqrt{0 + (1+i)(1-i) + (1)(1)}$$
$$= \sqrt{3}$$
$$\| u_3 \| = \sqrt{<u_3, u_3>} = \sqrt{(2)(2) + (1-i)(1+i) + (i)(-i)}$$
$$= \sqrt{7}$$

[$\because$ i. In C^3, if $u = (a, b, c)$, then $<u, u> = a\bar{a} + b\bar{b} + c\bar{c}$]

ii. The vector v is orthogonal to u_1 and u_2 if $<v, u_1> = 0$ and $<v, u_2> = 0$.

Now, $<v, u_1> = (1 - i)(1) + (-1)(-2i) + (1 - i)(-i)$
$= 1 - i + 2i - i - 1 = 0$

Also $<v, u_2> = 0$ (Student can check)

iii. Let $u = (a, b, c)$ be a vector which is orthogonal to both u_1 and u_3.

$\Rightarrow$ $<u, u_1> = 0, <u, u_3> = 0$

$\Rightarrow$ $a(1) + b(-2i) + c(-i) = 0$

and $a(2) + b(1 + i) + c(-i) = 0$

or $a - 2ib - ic = 0$ ...(1)

and $2a + (1 + i)b - ic = 0$ ...(2)

Solving (1) and (2), we get

$$\frac{a}{-3+i} = \frac{b}{-i} = \frac{c}{1+5i}$$

Thus, the vector orthogonal to both u_1 and u_3 is $u = (-3 + i, -i, 1 + 5i)$.]

Example 5.33. *Let $u = (1, 2, -1)$, $v = (2, 1, 4)$ and $w = (3, -2, -1)$ in R^3, then*

i. *show that they form an orthogonal set under the standard Euclidean inner product space for R^3 but not an orthonormal set.*

ii. *convert them into a set of vectors that will form an orthonormal set of vectors under the standard Euclidean inner product space for R^3.*

Solution. i. Given $u = (1, 2, -1)$, $v = (2, 1, 4)$, $w = (3, -2, -1)$, then

$$<u, v> = (1)(2) + (2)(1) + (-1)(4)$$
$$= 0$$

and
$$<v, w> = (2)(3) + (1)(-2) + (4)(-1)$$
$$= 0$$

Also,
$$\| u \| = \sqrt{1+4+1} = \sqrt{6} \neq 1$$

Hence the given vectors does not form an othonormal set.

ii. Now,
$$\frac{u}{\|u\|} = \frac{(1, 2, -1)}{\sqrt{1+4+1}} = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$$

and
$$\frac{v}{\|v\|} = \frac{(2, 1, 4)}{\sqrt{4+1+16}} = \left(\frac{2}{\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}\right)$$

and
$$\frac{w}{\|w\|} = \frac{(3, -2, -1)}{\sqrt{9+4+1}} = \left(\frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}\right)$$

Therefore, the set of orthonormal vectors are

$$\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right), \left(\frac{2}{\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}\right) \text{ and } \left(\frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}\right).$$

5.7 GRAM-SCHMIDT ORTHOGONALIZATION PROCESS

Statement: Every finite dimensional inner product space has an orthonormal basis.

Proof: Let $V(F)$ be an inner product space of dimension n.

Let
$$S = \{u_1, u_2, \ldots u_n\} \text{ be a basis of } V.$$

Since every orthonormal set is linearly independent and dim $V = n$, it is enough to construct an orthogonal basis of V.

Now, S being a basis is linearly independent.

$\therefore$
$$u_i \neq 0 \text{ for } i = 1, 2, \ldots, n$$

Let $v_1 = u_1$. Define $v_2 = u_2 - \dfrac{<u_2, v_1> v_1}{\| v_1 \|^2}$

Now
$$v_2 = 0$$

$\Rightarrow \qquad u_2 = \dfrac{<u_2, v_1> v_1}{\|v_1\|^2}$ = scalar multiple of v_1

or $\qquad u_2$ = scalar multiple of $u_1 \qquad [\because \quad v_1 = u_1]$

$\Rightarrow \quad (u_1, u_2)$ is linearly dependent

which is a contradiction, as $\{u_1, u_2\}$ being a subset of a linearly independent set S is linearly independent.

Thus $\qquad v_2 \neq 0$

Then
$$<v_2, v_1> = <u_2 - \frac{<u_2, v_1> v_1}{\|v_1\|^2}, v_1>$$
$$= <u_2 - \frac{<u_2, u_1> u_1}{\|u_1\|^2}, u_1> \qquad \{\because \quad v_1 = u_1]$$
$$= <u_2, u_1> - \frac{<u_2, u_1>}{\|u_1\|^2} <u_1, u_1> \qquad [\because \quad <u_1, u_1> = \|u_1\|^2]$$
$$\Rightarrow \quad <v_2, v_1> = <u_2, u_1> - <u_2, u_1>$$
$$= 0$$

$\Rightarrow \quad v_2$ is orthogonal to v_1 and so (v_1, v_2) is an orthogonal set.

Define
$$v_3 = u_3 - \frac{<u_3, v_1> v_1}{\|v_1\|^2} - \frac{<u_3, v_2> v_2}{\|v_2\|^2}$$
$$= u_3 - \sum_{i=1}^{2} \frac{<u_3, v_i> v_i}{\|v_i\|^2}$$

Again $\qquad v_3 \neq 0$ as $v_3 = 0 \quad \Rightarrow \quad \{u_1, u_2, u_3)$ is linearly dependent.

But $\{u_1, u_2, u_3\}$ being a subset of linearly independent set S is linearly independent.

Then
$$<v_3, v_1> = <u_3 - \frac{<u_3, v_1> v_1}{\|v_1\|^2} - \frac{<u_3, v_2> v_2}{\|v_2\|^2}, v_1>$$
$$= <u_3, v_1> - \frac{<u_3, v_1><v_1, v_1>}{\|v_1\|^2} - \frac{<u_3, v_2><v_2, v_1>}{\|v_2\|^2}$$
$$= <u_3, v_1> - <u_3, v_1> - 0 = 0 \quad [\because \quad <v_1, v_1> = \|v_1\|^2 \text{ and } <v_2, v_1> = 0]$$

In a similar way, we note that
$$<v_3, v_2> = 0 = <v_3, v_1>$$

Thus $\qquad <v_i, v_j> \neq 0$ where $i \neq j$ and $i, j = 1, 2, 3$

In this way, we can construct an orthogonal set $\{v_1, v_2, \ldots v_n\}$ of n vectors.

Take,
$$w_i = \frac{v_i}{\|v_i\|}$$

Then $\{w_1, w_2, \ldots, w_n\}$ is an orthonormal set of vectors in V.

Since an orthonormal set of vectors in an inner product space is a linearly independent set, so $\{w_1, w_2, \ldots, w_n\}$ is linearly independent. Also dim $V = n$, thus $\{w_1, w_2, \ldots, w_n\}$ forms an orthonormal basis of V.

SOME SOLVED EXAMPLES

Example 5.34. *Use the Gram-Schmidt orthogonalization process to transform the basis of R^3 generated by $u_1 = (1, 1, 1)$, $u_2 = (-1, 1, 0)$ and $u_3 = (1, 2, 1)$*

i. *into an orthogonal basis (v_1, v_2, v_3).*

ii. *into an orthonormal basis (w_1, w_2, w_3).*

Solution. Let $u_1 = (1, 1, 1)$, $u_2 = (-1, 1, 0)$ and $u_3 = (1, 2, 1)$, then

$$\| u_1 \|^2 = 3, \| u_2 \|^2 = 2, \| u_3 \|^2 = 6$$

and $<u_2, v_1> = 0, <u_3, u_1> = 4$ and $<u_3, u_2> = 1$

Define $v_1 = u_1 = (1, 1, 1)$

$$v_2 = u_2 - \frac{<u_2, v_1> v_1}{\| v_1 \|^2} = u_2 - 0$$

$\therefore$ $v_2 = u_2 = (-1, 1, 0)$

Also, $$v_3 = u_3 - \frac{<u_3, v_1> v_1}{\| v_1 \|^2} - \frac{<u_3, v_2> v_2}{\| v_2 \|^2}$$

$$= u_3 - \frac{4}{3} u_1 - \frac{u_2}{2}$$

$$= (1, 2, 1) - \frac{4}{3} (1, 1, 1) - \frac{1}{2} (-1, 1, 0)$$

$$= \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)$$

Hence the orthogonal basis is $\left\{(1,1,1), (-1,1,0), \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)\right\}$

ii. $$\| v_1 \| = \| u_1 \| = \sqrt{3}$$

$$\| v_2 \| = \| u_2 \| = \sqrt{2}$$

$$\| v_3 \| = \sqrt{<v_3, v_3>}$$

$$= \sqrt{\frac{1}{36} + \frac{1}{36} + \frac{1}{9}} = \sqrt{\frac{1}{6}}$$

Thus orthonormal basis $\{w_1, w_2, w_3\}$ is

$$\left\{\frac{v_1}{\| v_1 \|}, \frac{v_2}{\| v_2 \|}, \frac{v_3}{\| v_3 \|}\right\} = \left\{\frac{(1,1,1)}{\sqrt{3}}, \frac{(-1,1,0)}{\sqrt{2}}, \frac{\left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)}{1/\sqrt{6}}\right\}$$

Example 5.35. *Let V be a vector space over R of polynomials in R[x] of degree ≤ 2. Define an inner product on V as $<f(x), g(x)> = \int_0^1 f(x)\, g(x)\; dx$.*

If $\{1, x, x^2\}$ is a basis of V, find an orthonormal basis of V over R.

Solution. Here $\{1, x, x^2\}$ is a basis of V.

Let $u_1 = 1, u_2 = x, u_3 = x^2$

We apply Gram-Schmidt orthogonalization process to obtain an orthonormal basis of V over R.

Now, $\| u_1 \|^2 = \langle u_1, u_1 \rangle = \int_0^1 1 \,.\, 1 dx$

$$= [x]_0^1 = 1$$

Similarly, $\| u_2 \|^2 = \langle u_2, u_2 \rangle = \int_0^1 x \,.\, x \, dx$

$$= \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

and $\| u_3 \|^2 = \langle u_3, u_3 \rangle$

$$= \int_0^1 x^2 \,.\, x^2 \, dx = \left[\frac{x^5}{5}\right]_0^1 = \frac{1}{5}$$

Take $v_1 = u_1$

$\therefore$

$$v_2 = u_2 - \frac{< u_2, u_1 >}{\| u_1 \|^2} v_1$$

$$= x - \frac{\left[\int_0^1 x.1 \, dx\right] 1}{1}$$

$$= x - \left[\frac{x^2}{2}\right]^1 = x - \frac{1}{2}$$

Similarly, $\| v_2 \|^2 = \langle v_2, v_2 \rangle$

$$= \int_0^1 \left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right) dx$$

$$= \int_0^1 \left(x^2 - x + \frac{1}{4}\right) dx$$

$$= \frac{1}{12}$$

$$v_3 = u_3 - \frac{< u_3, u_1 > v_1}{\| v_1 \|^2} - \frac{< u_3, u_2 > v_2}{\| v_2 \|^2}$$

$$= x^2 - \frac{\left[\int_0^1 x^2.1 \, dx\right].1}{1} - \frac{\left[\int_0^1 x^2.\left(x - \frac{1}{2}\right) dx\right]\left(x - \frac{1}{2}\right)}{1/12}$$

$$= \left(x^2 - x + \frac{1}{6}\right)$$

Thus, $\| v_3 \|^2 = \langle v_3, v_3 \rangle$

$$= \int_0^1 \left(x^2 - x + \frac{1}{6}\right)\left(x^2 - x + \frac{1}{6}\right) dx$$

$$= \left[\frac{x^5}{5} - \frac{2x^4}{4} + \frac{4}{3}\frac{x^3}{3} - \frac{1}{3}\cdot\frac{x^2}{2} + \frac{1}{36}x\right]_0^1 = \frac{1}{180}$$

Put $w_1 = \frac{v_1}{\|v_1\|}, w_2 = \frac{v_2}{\|v_2\|}$ and $w_3 = \frac{v_3}{\|v_3\|}$

Hence, the required orthonormal basis is

$$\{w_1, w_2, w_3\} = \left\{1, \sqrt{12}\left(x - \frac{1}{2}\right), 3\sqrt{20}\left(x^2 - x + \frac{1}{6}\right)\right\}.$$

EXERCISE 5.4

1. Obtain an orthonormal basis with respect to standard inner product for the subspace of R^3 generated by (1, 0, 1), (1, 0, –1) and (0, 3, 4).
2. Show that the vectors $\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix}$ are orthogonal.
3. Let V be a vector space over R of polynomials in $R[x]$ of degree ≤ 2. Define an inner product on V as $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(x)\,g(x)\,dx$

   Where $f(x), g(x) \in V$. Find an orthogonal basis of V by using Gram-Schmidt orthogonalization process.
4. Construct an orthonormal set of vectors from the set $u_1 = (1, 2, 1)$, $u_2 = (2, 1, 4)$ and $u_3 = (4, 5, 6)$
5. Find the value of k so that the following expression form an inner product.

   $$(u, v) = u_1v_1 - 3u_1v_2 - 3u_2v_1 + ku_2v_2$$

   where $u = (u_1, v_1)$ and $v = (u_2, v_2)$ in R^2
6. Obtain a vector $V = (x, y, z) \in R^3$ so that V is perpendicular to (1, 0, 0) as well as (–1, 2, 0) with respect to the standard inner product.
7. Let $u = (1 + i, i, -1)$, $v = (1 + 2i, 1 - i, 2i)$, find $\langle u, v\rangle$ and $\langle v, u\rangle$.

   Is $\langle u, v\rangle$ equal to $\overline{\langle v, u\rangle}$? Also verify:

   i. $\langle u, v\rangle = \overline{\langle v, u\rangle}$ ii. $\langle u, v\rangle + \langle v, u\rangle = 2Re\,\langle u, v\rangle$

   iii $\langle u, v\rangle - \langle v, u\rangle = 2Im\,\langle u, v\rangle$

Answers

1. $\left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), (0,1,0)\right\}$

3. $\left\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{5}{8}}(3x^2 - 1)\right\}$

4. $\left\{\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(\frac{2}{\sqrt{93}}, \frac{-5}{\sqrt{93}}, \frac{8}{\sqrt{93}}\right), \left(\frac{7}{\sqrt{62}}, \frac{-2}{\sqrt{62}}, \frac{-3}{\sqrt{62}}\right)\right\}$

5. $k > 9$

6. $(0, 0, z), z \in R$

7. $\langle u, v\rangle = 2 + 2i$, $\langle v, u\rangle = 2 - 2i$

INTERESTING FACTS

- Deriving Special Relativity is more natural in the language of linear algebra. In fact, Einstein's second postulate really states that "Light is an eigenvector of the Lorentz transform." This document goes over the full derivation in detail. https://people.math.rochester.edu/faculty/chaessig/students/Adams(S10).pdf
- **Spectral Clustering**. Whether it's in plants and biology, medical imaging, business and marketing, understanding the connections between fields on Facebook, or even criminology, **clustering** is an extremely important part of modern data analysis. It allows people to find important subsystems or patterns inside noisy data sets. One such method is spectral clustering which uses the eigenvalues of a the graph of a network. Even the eigenvector of the second smallest eigenvalue of the Laplacian matrix allows us to find the two largest clusters in a network.
- **Dimensionality Reduction/PCA**. The principal components correspond the largest eigenvalues of $A'A$ and this yields the least squared projection onto a smaller dimensional hyperplane, and the eigenvectors become the axes of the hyperplane. Dimensionality reduction is extremely useful in machine learning and data analysis as it allows one to understand where most of the variation in the data comes from.
- **Low rank factorization for collaborative prediction**. Netflix does the prediction what rating you'll have for a movie you have not yet watched. It uses the SVD, and throws away the smallest eigenvalues of $A'A$.
- **The Google Page Rank algorithm**. The largest eigenvector of the graph of the internet shows how the pages are ranked.

VIDEO REFERENCES

Inner Product and Orthogonality

Gram Schmidt Orthogonalization

APPLICATIONS TO REAL LIFE

- The basic reproduction number (R_0) is a fundamental number in the study of how infectious diseases spread. If one infectious person is put into a population of completely susceptible people, then R_0 is the average number of people that one typical infectious person will infect. The generation time of an infection is the time, tG, from one person becoming infected to the next person becoming infected. In a heterogeneous population, the next generation matrix defines how many people in the population will become infected after timetGhas passed. R_0 is then the largest eigenvalue of the next generation matrix. That is how transmissibility of corona virus is being calculated.
- In image processing, processed images of faces can be seen as vectors whose components are the brightnesses of each pixel. The dimension of this vector space is the number of pixels. The eigenvectors of the covariance matrix associated with a large set of normalized pictures of faces are called eigenfaces; this is an example of principal component analysis. They are very useful for expressing any face image as a linear combination of some of them. In the facial recognition branch of biometrics, eigenfaces provide a means of applying data compression to faces for identification purposes. Research related to eigen vision systems determining hand gestures has also been made.

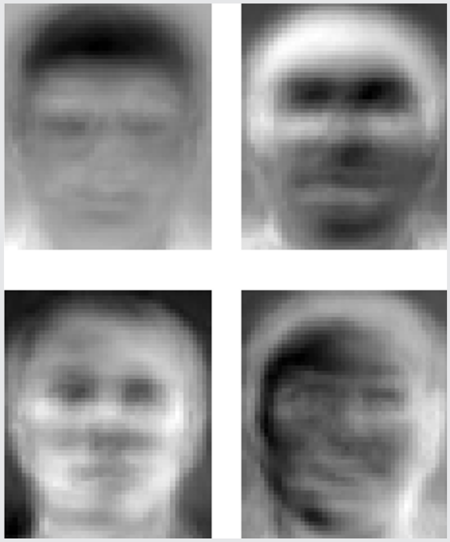

Fig. 5.1

- Similar to this concept, **eigenvoices** represent the general direction of variability in human pronunciations of a particular utterance, such as a word in a language. Based on a linear combination of such eigenvoices, a new voice pronunciation of the word can be constructed. These concepts have been found useful in automatic speech recognition systems for speaker adaptation.

SUBJECTIVE SOLVED QUESTIONS
(HOTS)

Example 1. *Consider 5 × 5 matrix* $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}$.

It is given that A has only one real eigen value, then find that real eigen value of A.

Solution. Characteristic equation is

$$|A - \lambda I| = 0$$

$$\therefore \qquad \begin{vmatrix} 1-\lambda & 2 & 3 & 4 & 5 \\ 5 & 1-\lambda & 2 & 3 & 4 \\ 4 & 5 & 1-\lambda & 2 & 3 \\ 3 & 4 & 5 & 1-\lambda & 2 \\ 2 & 3 & 4 & 5 & 1-\lambda \end{vmatrix} = 0$$

Operating $R_1 \to R_1 + R_2 + R_3 + R_4 + R_5$

$$\begin{vmatrix} 15-\lambda & 15-\lambda & 15-\lambda & 15-\lambda & 15-\lambda \\ 5 & 1-\lambda & 2 & 3 & 4 \\ 4 & 5 & 1-\lambda & 2 & 3 \\ 3 & 4 & 5 & 1-\lambda & 2 \\ 2 & 3 & 4 & 5 & 1-\lambda \end{vmatrix} = 0$$

Now, taking common

$$(15-\lambda)\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 5 & 1-\lambda & 2 & 3 & 4 \\ 4 & 5 & 1-\lambda & 2 & 3 \\ 3 & 4 & 5 & 1-\lambda & 2 \\ 2 & 3 & 4 & 5 & 1-\lambda \end{vmatrix} = 0$$

$$(15-\lambda)\,.\,|\text{Matrix}| = 0$$

$\Rightarrow \quad 15 - \lambda = 0$

$\Rightarrow \quad \lambda = 15$

$\therefore$ 15 is real eigen value of A.

Another approach

If sum of all rows or columns are same then that sum will be the eigen value of matrix.

In matrix A, Sum of all rows = 15

$\therefore$ 15 is a real eigen value of A.

Example 2. *Given that* $A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix}$ *and* $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *then what is the value of* A^3.

Solution. Given, $A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Characteristic equation is

$$|A - \lambda I| = \begin{vmatrix} -5-\lambda & -3 \\ 2 & 0-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (-5-\lambda)(-\lambda) + 6 = 0$

$\Rightarrow \quad 5\lambda + \lambda^2 + 6 = 0$

$\Rightarrow \quad \lambda^2 + 5\lambda + 6 = 0$

Since, every matrix satisfies its characteristic equation (Cayley-Hamilton Theorem)

$\therefore \quad A^2 + 5A + 6I = 0$

$\Rightarrow \quad A^2 = -5A - 6I \quad$ ...(1)

Multiply by A on both sides, we have,

$$A^3 = -5A^2 - 6AI$$

$$A^3 = -5(-5A - 6I) - 6AI \quad \text{(from (1))}$$

$$A^3 = 25A + 30I - 6AI$$
$$A^3 = 19 + 30I$$

Example 3. *A real 4 × 4 matrix A satisfies the equation $A^2 = I$, where I is 4 × 4 identity matrix. Find the positive eigen value of A.*

Solution. Since A satisfies the equation $A^2 = I$

∴ A is involutary matrix and eigen values of involutary matrix are ± 1

So, positive eigen value of $A = 1$.

Example 4. *Consider the matrix* $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, *where a and b are real number and b ≠ 0*

a. *Find all eigen values of A*

b. *For each eigen value of A, determine the eigenspace E_λ.*

c. *Diagonalize the matrix A by finding a non-singular matrix S and a diagonal matrix D such that $S^{-1}AS = D$.*

Solution. a. Characteristic equation of matrix A is

$$|A - \lambda I| = \begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (a-\lambda)^2 + b^2 = 0$$
$$\Rightarrow \quad (a-\lambda)^2 = -b^2$$
$$\Rightarrow \quad a-\lambda = \pm\, bi$$
$$\Rightarrow \quad \lambda = a \pm bi$$

Thus, eigen values of A are $a \pm bi$.

b. Eigen vectors w.r.t. eigen value $a + bi$

$$[A - (a+bi)I]X = 0$$

$$\Rightarrow \quad \begin{bmatrix} a-(a+bi) & -b \\ b & a-(a+bi) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} -bi & -b \\ b & -bi \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad -bix_1 - bx_2 = 0 \quad \Rightarrow \quad x_2 = -ix_1$$
$$bx_1 - bix_2 = 0 \quad \Rightarrow \quad x_1 = ix_2$$

General solution of system is

$$x_1 = ix_2$$

So, $$E_{a+ib} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Eigen space w.r.t eigen value $a - ib$

$$[A - (a-ib)I]X = 0$$

$$\Rightarrow \quad \begin{bmatrix} a-(a-ib) & -b \\ b & a-(a-ib) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} bi & -b \\ b & bi \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \qquad bix_1 - bx_2 = 0 \quad \Rightarrow \quad x_2 = ix_1$$

and $\qquad bx_1 + bix_2 = 0 \quad \Rightarrow \quad x_1 = -ix_2$

General solution of system is $x_2 = ix_1$

E_{a-ib} *i.e.*, eigen space w.r.t. eigen value $a - ib = \begin{bmatrix} 1 \\ i \end{bmatrix}$

c. Now, $\begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$ are *LI* and spans C^2, so, form eigenbasis for C^2.

$$\therefore \qquad S = \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$$

$$|S| = i^2 - 1 = -1 - 1 = -2$$

$$\therefore \qquad S^{-1} = \frac{1}{-2}\begin{bmatrix} i & -1 \\ -1 & i \end{bmatrix}$$

$$D = S^{-1}AS$$

$$= \frac{1}{-2}\begin{bmatrix} i & -1 \\ -1 & i \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$$

$$= \frac{-1}{2}\begin{bmatrix} ai-b & -bi-a \\ -a+bi & b+ai \end{bmatrix}\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} ai^2 - bi - bi - a & ai - b - bi^2 - ai \\ -ai + bi^2 + b + ai & -a + bi + bi + ai^2 \end{bmatrix} \Rightarrow \frac{-1}{2}\begin{bmatrix} -2(a+bi) & 0 \\ 0 & -2(a-bi) \end{bmatrix}$$

$$= \begin{bmatrix} a+bi & 0 \\ 0 & a-ib \end{bmatrix}$$

which is a diagonal matrix with eigen value $a + bi$ and $a - bi$.

Example 5. *Let A and B be $n \times n$ matrices. Suppose that A and B have the same eigenvalues $\lambda_1, ..., \lambda_n$ with the same corresponding eigen vectors $X_1, X_2, ..., X_n$ are linearly independent, then $A = B$.*

Solution. Suppose A and B have n L.I. eigen vectors $X_1, X_2, ..., X_n$, they are diagonalisable.

If we put $\qquad S = [x_1, ..., x_n]$.

Then S is invertible and we have

$$S^{-1}AS = D \text{ and } S^{-1}BS = D$$

Where D is diagonal matrix whose diagonal entries are eigenvalue as

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

It follows that $S^{-1}AS = D = S^{-1} BS$

and hence $A = B.$

Example 6. a. *Let u and v be the eigen vectors of A corresponding to the eigen values 1 and 3 respectively. Prove that u + v is not an eigenvector of A.*

b. *Let A and B be real matrices such that the sum of each row of A is 1 and the sum of each row of B is 2. Then show that 2 is an eigenvalue of AB.*

Solution. a. Given u and v are eigenvectors of A corresponding to the eigenvalues 1 and 3 respectively

So, $Au = 1u$

$Av = 3v$

Now, $A(u + v) = Au + Av$

$= 1u + 3v$

So, $(u + v)$ is not an eigenvector of A.

b. Try yourself.

Example 7. *Let A be a 3 × 3 real non-diagonal matrix with $A^{-1} = A$. Show that tr(A) = –det (A) = ± 1.*

Solution. Given, $A^{-1} = A \Rightarrow A^2 = I$

So, all the eigenvalues of A^2 are 1

Also, $A^2 - I = 0$

or, $(A - I)(A + I) = 0$

So, two eigenvalues of A are +1 and –1

Since, eigenvalues of A^2 are square of eigenvalues of A.

So, eigenvalues of A will either +1 or –1

So, the third eigenvalue can be +1 or –1

Determinant of matrix is equal to product of its eigenvalues

So, determinant can be ±1

If third eigenvalue is +1, $\text{tr}(A) = 1, \det(A) = -1$

If third eigenvalue is –1, $\text{tr}(A) = -1, \det(A) = 1$

So, $\text{tr}(A) = -\det(A) = \pm 1.$

SUMMARY

1. Let A be a square matrix of order n over a field F, if ∃ a non-zero column vector $X \in F^n$ such that $AX = \lambda X$ for some $\lambda \in F$, then X is called the Eigen vector of A corresponding to λ and λ is called an eigen value of A corresponding to X.
2. Sum of eigen values = Trace (A)

   Product of eigen values = det. (A)
3. A $n \times n$ matrix is diagonalizable iff it has n linearly independent eigen vectors.
4. Every finite dimensional inner product space has an orthonormal basis.
5. A vector $u \in V$ is said to be orthogonal to $v \in V$ if $\langle u, v \rangle = 0$
6. Two vectors u, v is said to be orthonormal if (*i*) $\langle u, v \rangle = 0$ (*ii*) norm of each vector u and v should be one.

OBJECTIVE QUESTIONS

1. Let $A = \begin{bmatrix} 2 & 0 \\ 3 & 5 \end{bmatrix}$ be expressed as $P + Q$, where P is symmetric matrix and Q is skew-symmetric matrix, which one of the following is correct?

a. $Q = \frac{1}{2}\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$ b. $\begin{bmatrix} 1/2 & -3/2 \\ 3/2 & 0 \end{bmatrix}$ c. $Q = \frac{1}{2}\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$ d. $Q = \begin{bmatrix} 0 & 3/2 \\ 3/2 & 0 \end{bmatrix}$

2. The columns of an orthogonal matrix form

a. an orthogonal set of vectors b. an orthonormal set of vectors
c. a linearly independent set d. All of the above

3. If $T : V \to V$ is a linear operator for dim $V = n$ and T has n distinct eigen values, then

a. T must be invertible b. T must be diagonalizable
c. T must be invertible as well as diagonalizable d. T is not diagonalizable

4. Let T be a vector space on $\mathbb{R}^2$ and let T be the linear transformation on V defined by $T(x, y) = (x + y, y)$. Then, the characteristic polynomial of T is

a. $1 - 3x + x^2$ b. $2 - 2x$ c. $1 - 2x + x^2$ d. $1 + x^2$

5. Let $T : C^3 \to C^3$ be defined by $T(x, y, z) = (x + y + z, -x - y, -x - z)$ and M be its matrix with respect to the standard ordered basis. The eigen values of M are

a. $-1, i, -i$ b. $1, i, -i$ c. $1, i, i$ d. $-1, -i, -i$

6. A matrix M has eigen values 1 and 4 with corresponding eigen vectors $(1, -1)^T$ and $(2, 1)^T$, respectively, then, M is

a. $\begin{bmatrix} -4 & -8 \\ 5 & 9 \end{bmatrix}$ b. $\begin{bmatrix} 9 & -8 \\ 5 & -4 \end{bmatrix}$ c. $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ d. $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$

7. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, then the modal matrix P is

a. $\begin{bmatrix} -1 & 1 \\ 1+i & 1-i \end{bmatrix}$ b. $\begin{bmatrix} 1 & 1 \\ 1-i & 1+i \end{bmatrix}$ c. $\begin{bmatrix} 1 & -1 \\ 1+i & 1-i \end{bmatrix}$ d. $\begin{bmatrix} -1 & -1 \\ 1-i & 1+i \end{bmatrix}$

8. If the characteristic roots of $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ are λ_1 and λ_2, then the characteristic roots of $\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$ are

a. $\lambda_1 + \lambda_2, \lambda_1 - \lambda_2$ b. $2\lambda_1$ and $2\lambda_2$ c. $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$ d. $\lambda_1 + \lambda_2$ and $|\lambda_1 - \lambda_2|$

9. If $A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which of the following is zero matrix?

a. $A^2 - A - 5I$ b. $A^2 + A - 5I$ c. $A^2 + A - I$ d. $A^2 - 3A + 5I$

10. The eigen values of the matrix $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ are

a. 1, 4 b. $-1, 2$ c. 0, 5 d. $2, -5$

11. Which one of the following is an eigen vector of the matrix

$$\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

a. $[1 \quad -2 \quad 0 \quad 0]^T$ b. $[0 \quad 0 \quad 1 \quad 0]^T$ c. $[1 \quad 0 \quad 0 \quad -2]^T$ d. $[1 \quad -1 \quad 2 \quad 1]^T$

12. The eigen vector of the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ are written in the form $\begin{bmatrix} 1 \\ a \end{bmatrix}$ and $\begin{bmatrix} 1 \\ b \end{bmatrix}$. What is $a + b$?

a. 0 b. 1/2 c. 1 d. 4

13. An eigen vector of $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ is

a. $[-1 \ 1 \ 1]^T$ b. $[1 \ 2 \ 1]^T$ c. $[1 \ -1 \ 2]^T$ d. $[2 \ 1 \ -1]^T$

14. Consider the following matrix $A = \begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}$. If the eigen values of A are 4 and 8, then

a. $x = 4, y = 10$ b. $x = 5, y = 8$ c. $x = -3, y = 9$ d. $x = -4, y = 10$

15. For the matrix $A = \begin{bmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{bmatrix}$, one of the eigen value is 3. The other two eigen values are

a. 2, –5 b. 3, –5 c. 2, 5 d. 3, 5

16. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, then the eigen values of A are

a. 2, 1, 0 b. 2, $(1 + i)$, $(1 - i)$ c. 2, –1, –1 d. 1, –1, 0

17. Which of the following matrix is not diagonalizable?

a. $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$ c. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

18. What is the norm of vector (–2, 3, 7)?

a. $\sqrt{60}$ b. $\sqrt{62}$ c. 7 d. $\sqrt{78}$

19. Let $u = (a_1, a_2)$ and $v = (b_1, b_2)$ be arbitrary members of R^2, then which of the following is not an inner product space on $R^2(R)$

a. $\langle u, v\rangle = a_1b_1 - a_2b_1 - a_1b_2 + 4a_2b_2$ b. $\langle u, v\rangle = a_1b_1 + a_2b_2$

c. $\langle u, v\rangle = a_1 + a_2 + b_1 + b_2$ d. $\langle u, v\rangle = a_1b_1 - 2a_1b_2 - 2a_2b_1 + 5a_2b_2$

Answers

1. c	**2.** d	**3.** b	**4.** c
5. a	**6.** d	**7.** b	**8.** c
9. c	**10.** c	**11.** b	**12.** b
13. b	**14.** d	**15.** b	**16.** c
17. c	**18.** b	**19.** c	

SUBJECTIVE UNSOLVED QUESTIONS
(HOTS)

1. Determine the values of a, b and c so that (1, 0, –1) and (0, 1, –1) are eigen vectors of the matrix

$$\begin{bmatrix} 2 & 1 & 1 \\ a & 3 & 2 \\ 3 & b & c \end{bmatrix}$$

2. Let P, D and A be real square matrices of same order such that P is invertible, D is diagonal and $D = PAP^{-1}$. If $A^n = 0$ for some $n \in N$, then show that $A = 0$.

3. Let A be an $n \times n$ real symmetric matrix with n distinct eigen values. Prove that there exists an orthogonal matrix P such that $AP = PD$ where D is a real diagonal matrix.

4. Find the number and exhibit all 2×2 orthogonal matrices of the form $\begin{bmatrix} 1/3 & x \\ y & z \end{bmatrix}$.

5. Find a 3×3 orthogonal matrix P whose first two rows are multiples of:

a. (1, 2, 3) and (0, –2, 3) b. (1, 3, 1) and (1, 0, –1)

6. Let A be a real skew-symmetric matrix, that is, $A^T = -A$. Then prove the following statements.

a. Each eigenvalue of the real skew-symmetric matrix A is either 0 or a purely imaginary number.

b. The rank of A is even.

7. Prove that if $A \in M_{n \times n}(F)$ has n distinct eigenvalues, then A is diagonalisable.

8. Prove that two distinct eigenvectors corresponding to the same eigenvalues are always linearly dependent.

9. For the given 2×2 matrix

$$A = \begin{bmatrix} a & b-a \\ 0 & b \end{bmatrix}$$

a. Find the eigen values of A:

b. For each eigenvalue of A, determine the eigenvector.

c. Diagonalise the matrix A.

d. Using the result of the diagonalisation, compute and simplify A^k for each positive integer k.

10. Let the set $\{v_1, v_2, ..., v_n\}$ be L.D. What happens when the Gram-Schmidt process of orthogonalisation is applied to it?

11. Let $W = (1, -2, -1, 3)$ be a vector in R^4. Find:

a. an orthogonal basis for $W^{\perp}$.

b. an ortho-normal basis for $W^{\perp}$.

12. Let $M = M_{2,2}$ with inner product $<A, B> = \text{tr}(B^T A)$. Find an orthogonal basis for the orthogonal complement of:

a. diagonal matrices

b. symmetric matrices

Answers

1. $[a = 2, b = 4, c = 4]$

4. $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}, \begin{bmatrix} a & b \\ -b & -a \end{bmatrix}, \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \begin{bmatrix} a & -b \\ -b & -a \end{bmatrix}$ where $a = \frac{1}{3}$ and $b = \frac{\sqrt{8}}{3}$

5. a. $\left[\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}; 0, \frac{2}{\sqrt{3}}, \frac{3}{\sqrt{13}}; \frac{12}{\sqrt{157}}, \frac{-3}{\sqrt{157}}, \frac{-2}{\sqrt{157}}\right]$

b. $\left[\frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}; \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}; \frac{3}{\sqrt{22}}, \frac{-2}{\sqrt{22}}, \frac{3}{\sqrt{22}}\right]$

9. When $a \neq b$

a. a, b

b. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

c. $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

d. $A^k = \begin{bmatrix} a^k & b^k - a^k \\ 0 & b^k \end{bmatrix}$

When $a = b$

a. a, a

b. $\begin{bmatrix} x \\ y \end{bmatrix}, x \neq 0, y \neq 0, x, y \in C$

c. $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

d. $A^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$

10. If V_1 is the first vector in the span of the earlier ones, then the corresponding vector in the process of orthogonalisation will be zero.

11. a. $u_1 = (0, 0, 3, 1), u_2 = (0, 5, -1, 3), u_3 = (-14, -2, -1, 3)$

b. $\frac{u_1}{\sqrt{10}}, \frac{u_2}{\sqrt{35}}, \frac{u_3}{\sqrt{210}}$

12. a. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

PROJECT/ ACTIVITIES/PRACTICAL

PROJECT

"Diagonalisation helps in determining powers of *A i.e. A* to the power *n*, where *n* is a general integer." Explain mathematically as well as with the help of an example.

ACTIVITY

Explain how a linear system of differential equations $(dx/dt) = X$, where X is a $m \times m$ diagonal matrix with constant entries, can be solved using the diagonalisation concept.

PRACTICAL

Implement the power method (with normalization), for computing eigen values and eigenvectors of a matrix $A \in R_{n\times n}$ in MATLAB.

KNOW MORE

1. If A is a (2×2) matrix over R with Det $(A + I) = 1 +$ Det (A), then we can conclude that

a. Det $(A) = 0$ b $A = 0$ c. Tr $(A) = 0$ d. A is non-singular

2. If $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$, then calculate A^9.

a. $511A + 510I$ b $309A + 104I$ c. $154A + 155I$ d. exp. $(9A)$

3. Let V be a vector space of real polynomials of degree atmost 2. Define a linear operator

$$T : V \rightarrow V \text{ by } T(x) = \sum_{j=0}^{i} x^j, i = 0, 1, 2$$

The dimension of the eigen space of T^{-1} corresponding to the eigen value 1 is,

a. 4 b. 3 c. 2. d. 1

4. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, then A^{50} is

a. $\begin{bmatrix} 1 & 0 & 0 \\ 50 & 1 & 0 \\ 50 & 0 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 & 0 \\ 48 & 1 & 0 \\ 48 & 0 & 1 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 0 & 0 \\ 24 & 1 & 0 \\ 24 & 0 & 1 \end{bmatrix}$

Answers

1. c **2.** a **3.** d **4.** c

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CO AND PO ATTAINMENT TABLE

Course outcomes (COs) for this course can be mapped with the programme outcomes (POs) after the completion of the course and a correlation can be made for the attainment of POs to analyze the gap. After proper analysis of the gap in the attainment of POs necessary measures can be taken to overcome the gaps.

Table for CO and PO attainment

Course Outcomes	Attainment of Programme Outcomes *(1- Weak Correlation; 2- Medium correlation; 3- Strong Correlation)*											
	PO-1	PO-2	PO-3	PO-4	PO-5	PO-6	PO-7	PO-8	PO-9	PO-10	PO-11	PO-12
CO-1												
CO-2												
CO-3												
CO-4												
CO-5												
CO-6												

The data filled in the above table can be used for gap analysis.